-9

MEDICI'S

RATIONAL MATHEMATICS.

SECTION A

GEOMETRY

PART I.

FIRST PRINCIPLES AND PRIMARY ELEMENTS

AND BY THE AUTHOR,
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1895.

MEDICI'S

RATIONAL MATHEMATICS.

SECTION A

GEOMETRY

PART I.

FIRST PRINCIPLES AND PRIMARY ELEMENTS

TAUGHT BY COMPASS AND RULER

ON THE

BLACKBOARD.

SOLD BY DEALERS IN TEXT-BOOKS,

AND BY THE AUTHOR,

CHAS. de MEDICI,

60 WEST 22ND STREET, NEW YORK CITY.

1895.

SECTION A.

Part I.

DEFINITIONS.

MATERIAL POINTS.

1. Pointed ends of anything, as, for example, a sharpened pencil point, or the points of compass-dividers, are material points.

PUNCTS.

2. A dent made by a material point in any material substance is called a punct. Hence, puncts are given *sites* of located material points.

DOTS.

• • •

3. Dots are symbolic marks for puncts, and represent located points.

DISTANCE.

A·   ·*B*

4. The intervening space between two located points marked by dots is called a distance. Hence the distance: *A B*.

LINES.

A ———— *B*

5. Lines are straight marks which represent distances. The line *A B* marks the distance between two given points represented by the dots *A* and *B*. Every true line marks the *shortest* distance between two points.

MATHEMATICAL POINTS.

6. Mathematical points may be grouped into three kinds: extreme points, points of unity and intersecting points.

EXTREME POINTS.

7. The location where a given distance ends is called an extreme point. Extreme points are represented by extremities of the line which marks the distance.

A———————*B* EXHIBIT I.

A and *B* are the lineal extremities of the line *A B*, which two extremities mark the two extreme points of a given distance represented by the line *A B*.

POINTS OF UNITY.

8. When two or more extreme points represented by a number of lineal extremities blend into one common point, the common point is called a *point of unity*.

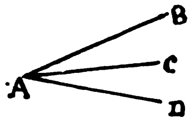

EXHIBIT 2.

A B, *A C*, *A D*, are three given lines. *A* marks the point of unity.

INTERSECTING POINTS.

9. Intersecting Points are produced when two lines cross each other.

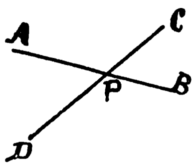

EXHIBIT 3.

The two lines *A B* and *C D* cross each other at the intersecting point *P*.

UNIFORM CURVES.

10. Uniform Curves are described by moving one extremity of a given line while the other extremity of the same line occupies a fixed location.

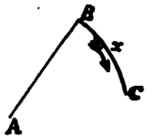

EXHIBIT 4.

A B is the given line. *A* represents the fixed location of the stationary extremity; *B* represents the moving extremity; *B x C* represents the curved course described by the motion of the extremity *B*. Curves described in this manner are uniform curves.

THE ENDLESS UNIFORM CURVE.

11. The endless uniform curve is a curve described from and to the self same point.

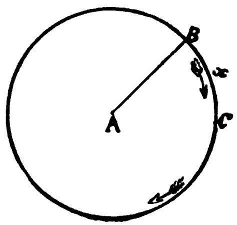

EXHIBIT 5.

It is produced by continuing the curve *B x C*, as described in Exhibit 4, until the starting point *B* is reached. This uniform endless curve marks the *longest* distance from and to the self same point, contrary to the straight line which marks the *shortest* distance between two given points. When drawn on a plane surface, the endless uniform curve is called a circle's circumference; but when the endless uniform curve is used to represent the greatest distance around a sphere or a globe, then it is called the periphery.

PRIME ELEMENTS.

12. The prime geometric elements are named:

dot line curve

By combinations of these prime elements the primary geometric forms are produced.

PRIMARY FORMS.

13. Geometric forms are represented by plane surfaces or by spaces bounded by geometric elements.

The primary forms are named: circles, segments and sectors.

THE CIRCLE.

14. The circle is a plane surface bounded by the circle's circumference which has already been described in Exhibit 5. The circle contains all the primary forms and every geometric form can be drawn in and evolved from the circle.

EXHIBIT 6.

THE SEGMENT.

15. The segment is formed by two elements: line and curve. Any line drawn in a circle from circumference to circumference forms a segment.

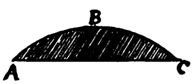

EXHIBIT 7.

The space bounded by the line *A C* and by the curve *A B C* represents a segment. When the curve of a segment is uniform, the segment is part of a circle. All segments which are parts of a circle are called *prime* segments

PRIME SECTORS.

16. The prime sector is formed by two lines of equal length and one uniform curve. The curve of the prime sector may be described with either of the two lines from a common point of unity which is located in a circle. Lines which form parts of a sector's boundery are called legs of the sector.

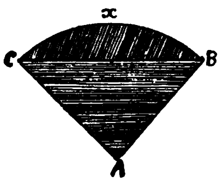

EXHIBIT 8.

The lines *A B* and *A C* together with the curve *C x B* form the prime sector *A B x C A*.

FIRST AXIOM.

17. *All prime segments and sectors are contained in and are parts of a circle.*

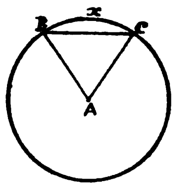

EXHIBIT 9.

For it is self evident that the sector *A B x C* is contained in a circle described by the line *A B* or by the line *A C*, and it is also self evident that the segment *C x B C* is contained in the circle as well as in the given sector which is a part of the circle.

PRINCIPAL ELEMENTS.

18. The principal elements are named: centers, radii, diameters, arcs, chords, angles and sines.

DEFINITIONS.

THE CENTER.

19. In any circle, the common point of unity for all the sectors in the circle is called the circle's center. This common point of unity must necessarily be *equidistant* from every part of the circle's circumference. since it is shown that the legs of sectors describe the circumference of circles. *C* marks the center in Exhibit 10.

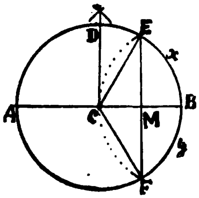

EXHIBIT 10.

RADIUS AND RADII.

20. Lines representing legs of sectors also mark and measure the distance from center to circumference of circles. Considered as a measure of distance between a given center and a given circumference, each line is called the radius; but when two or more are spoken of and named together, they are called radii. Exhibit 10 shows five radii, either of which is the circle's radius.

SECOND AXIOM.

21. *Every sector marks a center and a circle's radius.*

THE DIAMETER.

22. Two radii of any circle extending from the circle's center in opposite directions mark the longest distance in the circle. This, the longest line in the circle, is called the circle's diameter. Hence, every diameter equals two radii as *C A* and *C B*.

ANGLES.

23. Difference in direction of radii *within* a semicircle, or, the divergence of any two given radii in a semicircle, form what is

called an angle. Exhibit 10 shows the angles *A C D*, *A C E*, *D C E* and *F C E*.

ARCS.

24. Any part of a given circumference marked off (or measured) by a given angle is called an arc. Hence it follows, that every given arc measures an angle and every given angle measures an arc. Exhibit 10 and 11 show the two arcs *B x D* and *B y F* measured by the angles *D C B* and *B C F*.

CHORDS.

25. Any line which spans a given arc is called a chord. Exhibit 10 and 11 show the chord *E F* which spans the arc *F B E*. Now, since every arc and its chord form a segment, and as every segment and two radii form a sector, as shown by the figure *C E B F C*, it follows, that chords divide sectors into two parts: one, the segment *E F B E*, the other, the angular figure *C E F C* which is called the *angle-plane*. A line which divides the angle-plane into equal parts is called the plane's altitude, and a line which divides a segment into two equal parts is called the altitude of the segment.

SINES.

26. The altitude of an angle-plane and the altitude of a segment contained in a given sector, produce in conjunction with the chord which divides the sector, certain lines called *sines*.

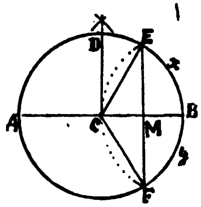

EXHIBIT 11.

That is: *E C F B E* is the given sector; *E F* is the chord which divides the sector's angle-plane from the segment; *M B* is the altitude of the sector's segment, and *M C* is the altitude of the

sector's angle-plane. *M C*, *M B*, *M E* and *M F* represent what are called sines.

THIRD AXIOM.

27. *All the principal elements are contained in the circle.*

SUB-ELEMENTS.

28. The sub-elements are evolved from the given principal elements and are named: right angles, obtuse angles, acute angles, perpendiculars, parallels, right sines, versed sines, co-sines, tangents and secants.

THE RIGHT ANGLE.

29. The right angle is obtained by dividing a semicircle into two equal parts, as shown in Exhibit 12, by the line *C D*, which divides the semicircle into the two equal parts: *D C A D* and *D C B E D*. Hence, the arc-measure for a right angle is one fourth of the circle's circumference.

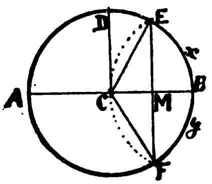

EXHIBIT 12

OBTUSE AND ACUTE ANGLES.

30. All arcs *greater* than one fourth of a circle's circumference measure *obtuse* angles, and all arcs *less* than one fourth of a circle's circumference measure *acute* angles. Hence, the angle *E C B* is acute, and the angle *E C A* is obtuse.

PERPENDICULARS.

31. Any two lines which form a right angle are said to be perpendicular to each other. Hence, the radius *C D* is perpendicular to the diameter *A B*, and *C D* is perpendicular to *C A*.

PARALLELS.

32. When two lines are perpendicular to a given third line, the two lines are said to be parallel.

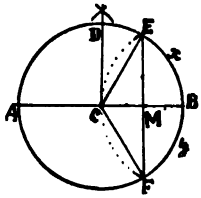

EXHIBIT 13.

The diameter *A B* is the given third line, which, with the other two lines (*C D* and *M E*) form right angles. Hence, *C D* and *M E* are parallel, because, they are both perpendicular to the common base *A B*.

THE RIGHT SINE.

33. The right sine is a given line which is perpendicular to one radius and parallel to another radius in the same circle. The line *E M* represents a right sine, because, it is perpendicular to the radius *C B* and it is parallel to the radius *C D*.

VERSED SINES AND CO-SINES.

34. "Versed sine" is the name for that distance of the radius which intervenes between the right sine and the circumference, as *M B*. (Exhibit 13.)

"Co-sine" is the name for the remaining distance of the radius intervening between the right sine and the circle's center, as *M C*. (Exhibit 13.)

TANGENTS AND SECANTS.

35. When a sine or a chord of one circle touches the circumference of another circle, such sine or chord is called a tangent.

Thus, the chord $M\,B$ of the greater circle over $A\ B$ is tangent to the minor circle over $O\ D$. (Exhibit 14).

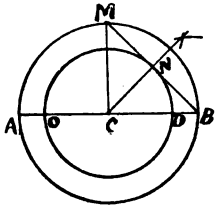

EXHIBIT 14.

When a line is drawn from the center of a minor circle and extended beyond the circumference, until it forms a point of unity with a tangent, such line is called a secant. Hence the secant $C\,M$.

FOURTH AXIOM.

36. *All geometric elements are contained in the circle.*

GEOMETRIC CONSTRUCTION.

37. Primary construction is effected by the use of ruler and compasses, within the compass of a circle.

38. Geometric construction is proven true when a common rule is given and applied to circles of different dimensions, and similar results are obtained.

39. The simplest process in geometric construction is called *bisecting*, which means: to divide anything given into two equal parts.

CONSTRUCTION OF THE RIGHT ANGLE

by the bi-secting process.

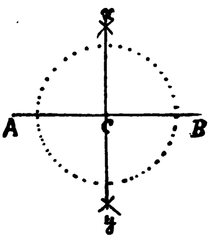

EXHIBIT 15.

40. $A\ B$ is a given line. With compasses find two intersecting points as x and y which shall be equidistant from the two extreme points A and B. Draw with the ruler the bisecting line $x\ y$ which divides the given line $A\ B$ into two equal parts as $C\ B$ and $C\ A$. This process, constructively demonstrates, that $x\ y$ is perpendicular to $A\ B$. For, when from the center C, with any radius less than one half the line $A\ B$, a circle is described, it is found, that the two lines $A\ B$ and $x\ y$ divide the circle into four equal parts. Hence, each of the four angles formed by the process is measured by one fourth of a given circumference, which is the measure for a right angle. (See § 29.)

HOW TO FIND THE CENTER OF ANY CIRCLE.

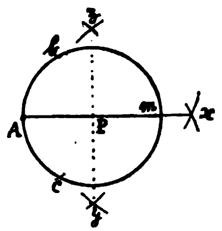

EXHIBIT 16.

41. When the circumference of a circle is given, mark a point anywhere in the circumference as at the point A. With compasses from A as a center, mark two equidistant points in the circumference as the points b and c. With compasses from b and c as centers, mark an intersecting point as x, equidistant from b and c; then draw a line from x to A which bisects the given circumference at m. Again, with compasses from A and m as centers, mark the equidistant points z and y and draw the line $z\ y$ which bisects the diameter $A\ m$ at P. Hence, P is the center sought.

THE THREE POINT PROBLEM.

42. Any three points not in the same line are contained in a periphery, which periphery can be found by compasses and ruler.

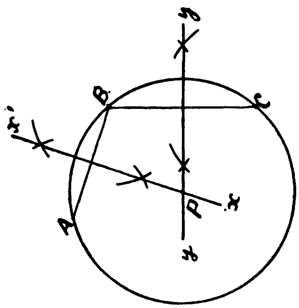

EXHIBIT 17.

A, *B*, and *C*, are three given points. Draw two lines which connect the given points, as the lines *A B* and *B C*. Bisect these connecting lines, shown by *y y'* and by *x x'*, and extend the bisecting lines until they intersect each other as shown by the intersecting point *P*. Now, from *P* as a center, describe a circle with a radius equal to the distance between either of the given points *A*, *B*, or *C* and the point *P*, then it is found, that a circle so described contains in its circumference the three given points.

FIFTH AXIOM.

43. *All geometric points are contained in the circle.*

SCHOLIUM.

(Explanatory Remarks.)

44. If all points not in the same line are contained in a circle's circumference, and if all points in one and the same line are contained in the *longest* line which necessarily must be a diameter of the *greatest* circle, it is self evident, that all geometric points are contained in the greatest circle. For, let the longest line be what it will, that line is still the generating factor of a circle's circumference greater in extent than itself and all-compassing as regards geometric forms and mathematical points.

HOW TO FIND THE CIRCLE OF WHICH AN ARC IS GIVEN.

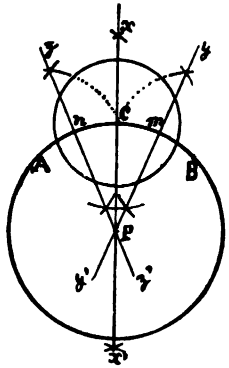

EXHIBIT 18.

45. *A C B* is a given arc. Bisect the arc *a c b* by the perpendicular line *x x'* which bisects at *c*. Describe a circle from *c* as a center over the given arc. Bisect the two arcs *b m c* and *c n a* by the lines *y y'* and *z z'* which produce the intersecting point *p*. The point *p* is the center sought and *p m* and *p n* are radii in the circle of which *A C B* is a given arc.

HOW TO CONSTRUCT PARALLEL LINES WITH COMPASSES.

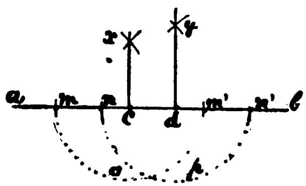

EXHIBIT 19.

46. On the given base line *a b* two points are given: *c* and *d*. It is required to construct two lines at the points *c* and *d* parallel to each other and perpendicular to the common base line *a b*. From *c* as a center with any radius, describe a semicircle as *m o m'* and construct the perpendicular *c x* according to rules given in § 30, and § 31. From *d* as a center with any radius, describe a semicircle as *n p n'*. Proceed as before, and construct the second perpendicular *d y*. Then the two lines *c x* and *d y* are proven parallel, since both are perpendicular to the same base line. (See § 32.)

HOW TO PROVE SLANTING LINES PARALLEL OR NOT PARALLEL WHEN NO BASE LINE IS GIVEN.

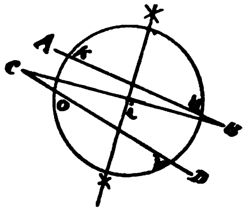

EXHIBIT 20.

47. If two lines as A B and C D are given without a base line and it is required to prove that these two lines are or are not parallel, draw a line between the two extremities B and C and bisect that line at the point i. From i as a center, with a radius less than one half the line B C, describe a circle, then if the arcs o k and u v are equal, the lines a b and c d are parallel, but when the arcs o k and u v are unequal, as in the Exhibit 20, the lines A B and C D are not parallel.

EVOLUTION OF SECTORS AND SINES BY THE USE OF COMPASSES.

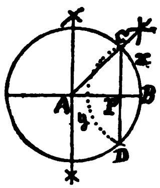

EXHIBIT 21.

THE OCTANT.

CONSTRUCTION.

48. Describe a circle. Quarter the circle. Bisect the quadrant. Then the octant A B x C A is given. With compasses from the point B as a center, and with the octant chord B C for radius, describe the curve $C y D$. Draw the quadrant-chord C D which intersects the radius A B at P, then three sines are given as a result of the operations: the right sine C P, the versed sine $P B$, the co-sine $P A$.

THE SEXTANT.

49. The sextant is a sector equal to one sixth of a circle.

MANNER OF CONSTRUCTION.

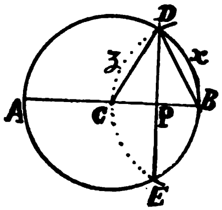

EXHIBIT 22.

Describe a circle. Draw the diameter and lay off with radius, from extreme of diameter, a chord equal to the radius. Then the sextant arc *B x D* and the sextant chord *B D* are given. Draw the radius *C D*, then the sextant *C B x D C* is given. With *B D*, as a radius, from *B* as a center, continue the curve *D z C* to *E*, and draw the tridrant-chord *D E* which intersects the radius *C B* at *P*. By these operations, three sines are given: *D P*, the right sine of a sextant, *P B*, the versed sine of a sextant, and *P C*, the co-sine of a sextant.

THE TRIDRANT.

50. The tridrant is a sector equal to one third of a circle.

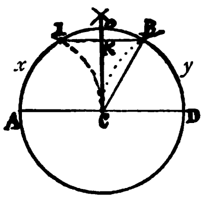

EXHIBIT 23.

CONSTRUCTION.

Draw a circle and mark off with the radius a sextant arc as *D y B*. Draw the radius *C B*, then a tridrant is given and described by the figure *A C B x A*. Now, if the tridrant-arc *B x A* is bisected at *I* and a sextant-chord is drawn as *B I*, and when the sextant arc

$B\ O\ I$ is bisected at O, and the radius $O\ C$ is drawn perpendicular to $A\ D$, then three sines are given: the right sine $B\ K$, the versed sine $O\ K$ and the co-sine $K\ C$, all of which are contained in the sector $B\ C\ O\ B$ which represents one twelfth of a circle. The tridrant-chord of the sector $A\ C\ B\ I\ A$ is represented by drawing a line from A to B.

SCHOLIUM.

51. Construction of quadrants and semicircles produces no sines, but the quadrant represents the sum of all sines, and the semicircle represents the sum of all angles and sectors. The greatest sector is the semicircle less the least sector and the least sector is the difference of the greatest sector and the semicircle, which the following illustration shows.

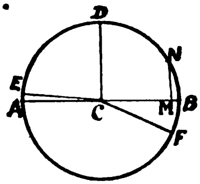

EXHIBIT 24.

If $A\ C\ E$ represents the least acute angle, then $E\ C\ B$ represents the greatest obtuse angle. And it follows, since the greatest and the least of all angles are contained in the semicircle, no arc greater than the semicircumference is required for the measurement of any and all angle-planes of sectors. Therefore, it is not proper to employ the term sector to parts of a circle greater than the semicircle. Thus, "section" $A\ C\ F\ B\ D\ E\ A$ is the better term in cases where a certain portion of a given circle is greater than the semicircle as shown in the Exhibit.

If the versed sine $M\ B$ represents the least sine, then the co-sine $M\ C$ represents the greatest sine. Now, as the radius of the circle is shown to be the sum of the least and the greatest sines, it follows, that the quadrant of any circle contains all the sines which can occur in geometric construction.

THE GEOMETRIC DIAL.

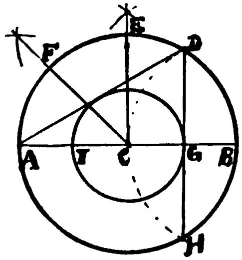

EXHIBIT 25.

52. The geometric Dial represents the summary of the 27 elements which comprise the alphabet of mathematics.

SUMMARY OF PART I, SECTION A.

THREE GEOMETRIC POINTS:

Extreme Points—Points of Unity—Intersecting Points.

THREE PRIME ELEMENTS:

Dot—Line—Curve.

THREE PRIMARY FORMS:

Circle—Sector—Segment.

EIGHT PRINCIPAL ELEMENTS:

Periphery (Circumference) — Center — Radius — Diameter. Angle—Arc—Chord—Sines.

TEN SUB-ELEMENTS:

Right Angle—Obtuse Angle—Acute Angle—Right Sine. Versed Sine — Co-Sine — Perpendiculars — Parallels — Tangents—Secants.

FIVE PRIMARY SECTORS:

Semicircle—Tridrant—Quadrant—Sextant—Octant.

FIVE BASIC AXIOMS.

THE THREE POINT PROBLEM

THE GEOMETRIC DIAL.

SUPPLEMENT.

TUTORS' SCHOLIUM.

(SECTION A, PART I.)

Great Responsibility rests with the tutor who imposes on himself the task of teaching Geometry as it should be taught. The new methods of teaching elementary geometry as presented in this book are designed to popularize the science. One object aimed at is to make the study interesting to the pupil by proper object-teaching which addresses the understanding and elicits inquiry, thereby to avoid taxing the memory with information not understood. The tutor should not trust to definitions expressed by language alone. Every verbal definition should be exemplified by some tangible object, some self executed demonstration or some operative process. From first to last, such objects, demonstrations and processes are formulated in this book in a concatenated series of orderly evolved issues, which, step by step introduce and explain the necessary technical terms, without wasting time in memorizing anything until it is required for use.

A special new feature of this Geometry is, that all geometric construction is effected *within* a circle, which keeps constantly the fact clear in the mind of the pupil, that every part of any geometric form whatever is a geometric element in some way related to the circle.

MEDICI'S

RATIONAL MATHEMATICS.

SECTION A

GEOMETRY

PART II.

FIRST PRINCIPLES OF COMMENSURATION,

FOUNDED ON

NATURAL DIVISION AND INHERENT DIMENSIONS
OF GEOMETRIC ELEMENTS,

TAUGHT BY THE USE OF COMPASSES AND RULER.

PART III.

CLASSIFICATION OF GEOMETRIC FIGURES AND FORMS,

With Analytic and Synthetic Aspects of
Component Parts

THIS PART OF GEOMETRY SHOULD BE TAUGHT

IN OBJECT - LESSONS

By the Aid of True Representative Devices constructed
for that purpose.

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AND BY THE AUTHOR,

CHAS. de MEDICI,

60 WEST 22ND STREET, NEW YORK CITY.

1895.

PREFACE.

(ADDRESSED TO THE TUTOR.)

ORIGIN OF GEOMETRY.

The first conception of Geometry was created in man's mind by observing a common phenomenon which can be observed by any one living now. The event dates back to the imagined primitive man who found himself on an island surrounded by water with no other limit than that where sea and sky mingles in one unbroken, endless and uniform curve. Now-a-days we speak of that curve as the compassing horizon, and when the same curve is depicted in a miniature diagram we call it the circumference of a circle. Studying his situation, the first student found himself posing in the middle of a circle. He occupied the position there, which now we know by the name of center. Instinct of imagination created a desire to imitate and describe what he had observed. He found a forked stick and fixed one prong in the ground where he had stood, to represent himself, and with the other, he described the endless uniform curve he had observed. Thus, the first crude draft of a geometric diagram was done; the first lesson in Elementary Geometry had been taught by nature without the first student's knowledge of geometry as we know it now; yet, the untutored pupil successfully defined and described to himself the elements now known to us as center, radius and circumference of the circle, by simply imitating a natural phenomenon he observed, and by using a natural instrument (the forked stick) which later led to the construction of the compass-divider. The second lesson, given by nature, occurred when the rising sun appeared to describe a curve overhead similar to the one observed in the horizon. This new phenomenon, in connection with the former, created the idea of a circle and a semicircle at right angles.

PREFACE.

But, of course, the names of these things, the first geometer knew not. His inventive genius, however, stirred his mind to invent a geometric toy in imitation of the semicircumference traversed by the sun. He bent a willow reed in imitation of the solar transit and fastened the ends of the reed in the drawn circle's circumference directly in line with the central prong. This operation led to the idea of the longest line (the diameter) in a circle. Next, he may have picked up on the beach a star shapen shell resembling the radiating sun, and may be, he entertained himself by sliding the star shapen shell along the curved reed to represent the motion of the sun. The radiating diverging rays from the sun, gave the first idea of angles and angular figures, and it is possible, that the primitive geometer represented the angles thus seen, by placing reeds in position of chords, sines and secants. Fired with enthusiasm, common to all discoverers, he determined to preserve and transmit to posterity what he had found. He reduced to diagram whatever he studied out from nature, and in due time he, or some one else, constructed in the circle all the twenty-seven elements which to this day constitute the geometric alphabet and which are represented in the geometric dial, Exhibit 25, Part 1, Section A, Page 17.

This sketch of possibilities which may have given birth to the science of geometry suggests, that geometry is of divine origin and cannot be changed in principle by arbitrary methods.

Section A

PART II

SECTION A.

Part II.

TRISECTING,

1. Trisecting means to divide something given into three equal parts as bisecting means to divide something given into two equal parts.

TRISECTING THE CIRCLE.

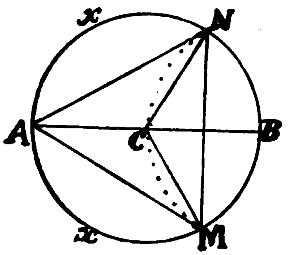


EXHIBIT I.

2. The three trisecting points in any circle's circumference are obtained as follows. Draw the diameter $A\ B$ and mark the center C. Describe the curve $N\ C\ M$ from B with the radius $B\ C$. Draw the lines $N\ C$ and $C\ M$, then it is found that three sectors are given, viz: $A\ x\ N\ C\ A$, $N\ B\ M\ C\ N$, $M\ x\ A\ C\ M$, each of which is a tridrant and equals one third of the given circle. When the three tridrant-chords are drawn as, $N\ M$, $M\ A$ and $A\ N$, then the segments and the angle-planes of the tridrants are also given.

TRISECTING THE SEMICIRCLE.

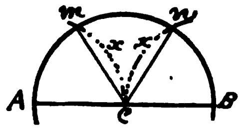


EXHIBIT 2.

3. A semicircle is trisected when the semi-circumference is divided into three equal parts.

The lines *C A* and *C B* are two given radii forming the diameter *A B*. From *A* with *A C*, and from *B* with *B C*, describe the arcs *C x m* and *C x n*; these arcs mark the intersecting points *m* and *n* which also are the trisecting points of the arc *A m n B*.

TRISECTING THE QUADRANT.

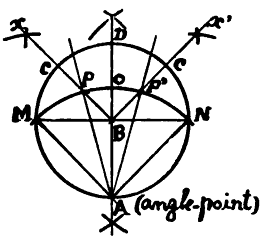


EXHIBIT 3.

4. *A N O M A* is a given quadrant, *M O N* is the quadrant's arc, *M B N* is the quadrant's chord. Bisect the quadrant by the line *A D*. Describe a circle over the chord *M N* from the bisecting point *B*. Bisect the two arcs *M C D* and *N C D* by the lines *B x* and *B x'*, then the two intersecting points (*P*, *P'*) are given, which divide the given arc *M O N* into three equal parts. Next, draw lines from *P* to *A* and from *A* to *P'*, then the given quadrant *A M O N A* is trisected, as show by the sectors *P' A N*, *P A P* and *P A M*.

THE TRISECTING FACTORS.

5. The trisecting factors are: an angle, a curve and a line.

The angle formed by the sector *A B C*, in the exhibit 3, is called the trisecting angle and is measured by *three eights* of a circle's circumference.

6. This angle, alone, is sufficient for trisecting any quadrant and its arc, as shown in exhibit 3; but if any other sector is given the arc of which is less than a quadrant arc, or when a sector is given, the arc of which is greater than a quadrant arc, then, a trisecting *curve* and a trisecting *line* are required to complete the operation.

7. The trisecting *curve* is the arc of a quadrant described over the chord of the arc given for trisection. The trisecting curve is obtained by describing a circle over the chord of the given arc, which circle must be quartered in order to obtain the angle-point, as shown in exhibit 4.

8. The trisecting *line* is a line drawn from the angle-point of the constructed quadrant to the point where the trisecting angle intersects the trisecting curve. If the trisecting curve falls below the given arc the line is extended until it intersects the given arc and marks the trisecting point. If the trisecting curve rises above the given arc the trisecting line in its course intersects the given arc at a point which marks one third of the arc.

DEMONSTRATIONS.

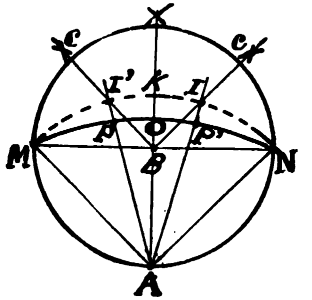


EXHIBIT 4.

9. Exhibit 4 shows the given arc $M\ O\ N$, which is less than a quadrant arc, in which case, the trisecting curve $M\ K\ N$ rises above the given arc $M\ O\ N$. The trisecting angle $A\ B\ C$ cuts the trisecting arc $M\ K\ N$ at the point I, and when the trisecting line $A\ I$ is drawn, the point P on the given arc $M\ O\ N$ is found to be the trisecting point.

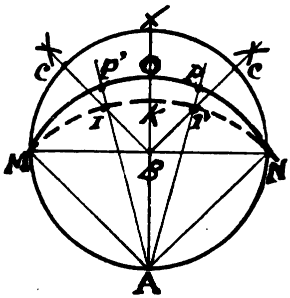


EXHIBIT 5.

10. Exhibit 5 shows the arc $M\ O\ N$, which is greater than a quadrant arc, in which case, the trisecting curve $M\ K\ N$ falls below the given arc $M\ O\ N$. The trisecting angle $A\ B\ C$ cuts the trisecting arc $M\ K\ N$ at the point I and when the trisecting line $A\ I$ is extended to the intersecting point P on the given arc $M\ O\ N$, then the required trisecting point is given at P.

TRISECTING A GIVEN LINE.

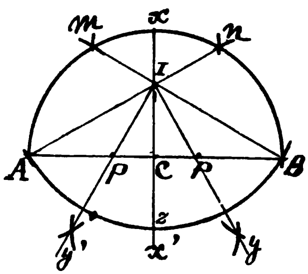


EXHIBIT 6.

11. $A\ B$ is the given line. Bisect $A\ B$ at C by the line $x\ x'$ Construct a semicircle on the given line and trisect that semicircle's

arc at m and n. Draw lines from A to n and from B to m and mark the intersecting point I. This intersecting point is found to be the center of a circle in which the given line $A\ B$ is a tridrant chord. Draw the tridrant arc $A\ z\ B$ and bisect the arcs $B\ z$ and $z\ A$ by the lines $y\ I$ and $I\ y'$, then, two trisecting points, P and P, are marked on the given line $A\ B$.

COMMENSURATIONAL DIVISION.

(Described.)

12. Commensurational division of geometric elements involves two fundamental principles: the principle of equality and the principle of equity. The principle of equality is shown when a line is divided by a common measure into equal parts. Constructively this is effected by compasses. The same principle is shown, when two given lines are each divided into an equal number of parts, by *sameness* in the number of parts, or, by sameness in the parts themselves.

The principle of equity, on the contrary, is identified by *difference* in the parts, in the number of the parts, or in both the number and the parts. But, whatever the difference is, it must be known and defined, and any such difference defined and expressed, is called: *proportional* difference.

DEMONSTRATION BY RULER AND COMPASSES.

EQUABLE DIVISION.

$$\begin{array}{ccccccc} & B & & C & & & \\ A\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot D \end{array}$$

13. $A\ C$ is a given line trisected with the common measure $A\ B$. By the same measure the line $A\ C$ can be extended to D, so that the line $C\ D$ equals the given line $A\ C$; that is to say, either line, $A\ C$ or $C\ D$, are composed of three $A\ B$. Hence, $A\ C$ and $C\ D$ are commensurately related by the commensurate measure $A\ B$.

In this case there is no proportional difference apparent, since *A C* and *C D* are equal and both are measurable by the common measure *A B*. This kind of division is called equable division.

EQUITABLE DIVISION.

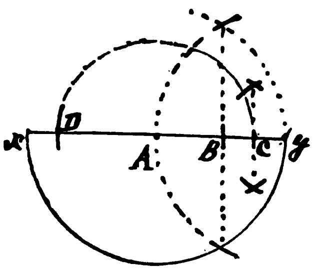


EXHIBIT 7.

14. If three lines are given as *C D*, *x y*, and *C x*, it is self-evident that neither one of these lines are a common measure for either of the other two. Hence, there is a proportional difference to be found. To find this difference, make the longest given line the diameter of a circle in which *A* marks the center. From the center *A*, describe with *A C* for radius, another semicircle, and bisect the greater radius *A y* at *B*. Then it is found that *C y*, or the difference of *x y* and *C x*, is a common measure for all the given lines, since *C y* equals *x D* and *D C* equals *B x*.

By the common measure *C y* it is shown that the longest given line *x y* contains eight *C y*, it is shown that the given line *x C* contains seven *C y*, while the shortest given line *C D* contains six *C y*. Hence, a proportional difference of one *C y* is obtained.

This kind of division illustrates what is called equitable division, and it shows that proportional differences are adjustable by common measures.

SCHOLIUM.

15. Any dividing measure wich divides a given quantity into equal parts without leaving a remaining fraction is called an *aliquot* divisor. The quantity divided is called the dividend and the number of times the divisor is contained in the dividend is called the quotient. If a dividend is given and the divisor (or dividing measure) is such that a fraction is left which the divisor can not divide, such divisor is called *aliquant*, and the quotient and the remaining fraction together are called a surd value. Exhibit 7 shows that the line $x\,y$ is a surd value when measured by the line $C\ D$. That is, the dividend $x\,y$, measured by the divisor $D\ C$ gives the quotient: one $D\ C$ and a fraction equal to the line $B\,y$. Exhibit 7 also shows a number of different proportions, namely: $B\ C$ is proportioned to $B\ A$ as 1 to 2; $A\ D$ is proportioned to $A\ B$ as 3 to 2; $B\ C$ is to $A\ D$ as 1 to 3; $B\ C$ is to $A\,y$ as 1 to 4.

MATHEMATICAL EQUITY.

16. The first rule of equity applies to given lines of different lengths divided into an equal number of parts justly proportioned to the lengths of the given lines.

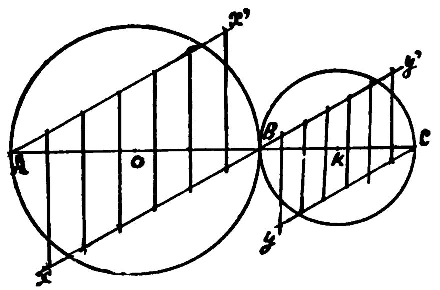


EXHIBIT 8.

(Demonstrate by Ruler and Compasses.)

$A\ B$ and $B\ C$ are two given lines of different lengths, each of which shall be divided into seven equal parts. With compasses bisect the lines $A\ B$ and $B\ C$ at O and at K. From O as a center,

describe with O B as a radius, the major circle, and from K as a center, describe with K B as a radius, the minor circle. Construct in each of these circles two parallel chords of equal arcs, and extend these chords indefinitely as shown by the lines $x\ y'$, $A\ x'$ and $C\ y$. By trial with compasses find an approximate measure slightly greater than one seventh of the given line. On the several chords and their extension lay off six of these trial measures (or one less in number than the number required) and mark the dividing points. Now connect these dividing points by ruler, then it is found that the ruler divides the given lines A B and B C into the required seven parts. Each of the minor parts are evidently proportioned to each of the major parts as the minor line B C is proportioned to the major line A B.

17. Hence the Axiom: *Unequal quantities each divided into the same number of parts are proportioned each to the other, as are the parts which divide the quantities.*

18. The second rule of equity applies to commensurational division of a circle's circumference when the circle's diameter is divided into a given number of equal parts.

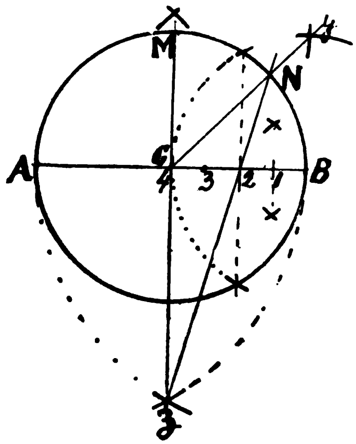


EXHIBIT 9.

Let A B represent a given circle's diameter divided into eight equal parts marked by the points 1, 2, 3, 4 etc. From A and B as centers, and with the diameter A B as radius, describe two curves which intersect at the point z outside the circle. This point is technically called the commensurational point and is equidistant from the extremities of the given diameter. The commensurational point is also the center of another circle in which the given diameter is a sextant's chord. From the point z draw a line to the given circle's circumference at N through the *second dividing point* in the given diameter. Then it is found, that the point N is the bisecting point in the quadrant arc B N M, and consequently, the arc B N is proportioned to the whole of the given circle's circumference as one of the parts (B 1) of the diameter is proportioned to the whole diameter.

19. The third rule of equity applies to commensurational division of a circle's diameter when the circumference is divided into a given number of equal parts.

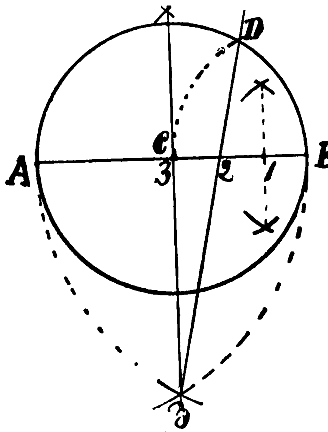


EXHIBIT 10.

A B is the diameter of a given circle the circumference of which is divided into six equal parts, one of which is shown by the arc B D. From the commensurational point z draw the line z D

through the point (2) on the diameter. Bisect the distance (B 2) at 1, then it is found that the radius $B\ C$ is divided into three equal parts, one of which is proportioned to the whole diameter as the arc $B\ C$ is proportioned to the whole circumference.

DIFFERENTIAL DIVISION.

20. Geometric elements cannot be commensurately related until they are divided into certain parts fitted to the relation. Hence, geometric lines have no fixed numeric value (number of parts) but must be differently divided when by change of position the line becomes an element of a different form. Exhibit 11 explains this differential principle which is part of geometric equity.

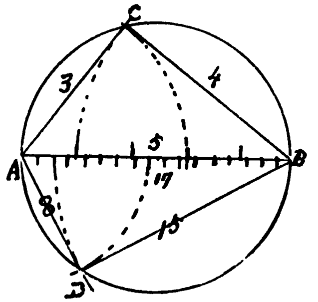


EXHIBIT 11

When a given diameter as $A\ B$ is divided into five equal parts, and the two chords $A\ C$ and $C\ B$ are drawn, these two chords measure respectively four and three of the diameter's five parts. Hence, a common measure is given for all three lines: $B\ C$, $C\ A$ and $A\ B$. Now it is found, if two other chords as $B\ D$ and $D\ A$ are given, these two chords are not divisible by the former common measure (one fifth) of the diameter $A\ B$. But, if $A\ D$ is divided into eight equal parts, each of these parts is a common divisor for both $A\ B$ and $D\ B$ which measure respectively, seventeen and fifteen of the given parts. Hence, it is shown, that the fitting measure for $B\ D$ and $D\ A$ is one seventeenth part of $A\ B$, while the fitting measure for $B\ C$ and $C\ A$ is one fifth of $A\ B$.

EVOLUTION OF NATURAL CHORDS, SINES AND ARCS.

(By Division.)

The first of the natural chords is called:

THE CUBE CHORD.

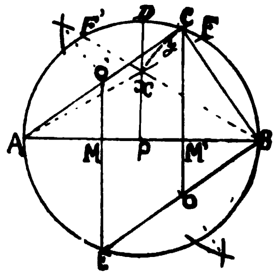


EXHIBIT 12.

21. The cube-chord $A\ C$, is obtained by trisecting a semi-circumference at the points F, F', and by drawing lines from F to A and from F' to B, which lines intersect at the point x on the perpendicular radius $P\ D$. Now if from B as a center, and with $B\ x$ as a radius, the arc $x\ y\ C$ is described, then, a line drawn from C to A marks the cube-chord $C\ A$.

PROPERTIES OF THE CUBE-CHORD.

22. The cube-chord is a factor in trisecting the diameter. For if two cube-chords as $B\ E$ and $A\ C$ (Exhibit 12) are drawn parallel in the same circle and both chords are bisected at O and O', and if lines are drawn from O to C and from O' to E, perpendicular to $A\ B$ and parallel to $P\ D$, then the given circle's diameter is trisected at the points M, M'.

SCHOLIUM.

23. *When the cube-chord is made the edge of a cube, the bulk of the cube equals the bulk of a globe constructed over the circle in which the chord is drawn.*

CONSTRUCTIVE DEMONSTRATION.

(By Ruler and Compasses.)

24. The process of evolving the natural chords, arcs and sines by construction, is exemplified in exhibit 13 and explained in the following paragraph.

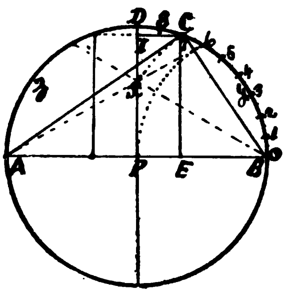


EXHIBIT 13.

Construct in a given quartered circle the cube chord *A C*, which produces three arcs: *B y C*, *C 8 D* and *C D z A*. Draw the lesser chord *C B* and construct the sines *C E* and *C I*. Then, the versed sines *B E* and *D I*, and the co-sines *P E* and *P I* are also produced. Hence it is said, that these several arcs, sines and chords are *natural*, because they are, as it were, evolved in consequence of the construction of the cube chord. Every other chord drawn in a circle evolves a number of these natural elements.

SCHOLIUM.

25. By commensurational division of the semicircumference into eighteen equal parts (arcs) it is found that the point *C*, which is one extremity of the cube chord *A C*, marks on the semicircumference, seven arcs plus a fraction, and eleven arcs minus a fraction. This common fraction is just one twentieth part of an arc when the circumference is divided into thirty-six equal arcs.

26. The second of the natural chords is called:

THE RECTIFYING CHORD.

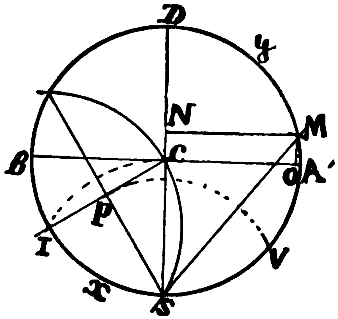


EXHIBIT 14.

The rectifying chord spans two sevenths of the circumference. The rectifying chord *S M* is obtained by quartering a circle and marking the sextant *S x I C S*, which sextant gives the right sine *S P*. Now, when the sine *S P* is used as a dividing chord for the circle's circumference, it marks from *S* to *V* one seventh of the circumference.

By doubling the arc *S V* the arc *S V M* is given and is shown to be two sevenths of the circumference. Hence, *S M* represents the rectifying chord as defined.

The arcs evolved by the construction of this chord are: *D y M*, *M A*, and *M V S*. The sines are: *M O*, *O A*, *O C*, and *M N*, *N C*, *N D*.

SCHOLIUM.

27. *The rectifying chord equals in extent one fourth of the circle's circumference, hence, four of these chords equal the rectified circumference, which explains the meaning of its name.*

The third of the natural chords is called:

THE SQUARING CHORD.

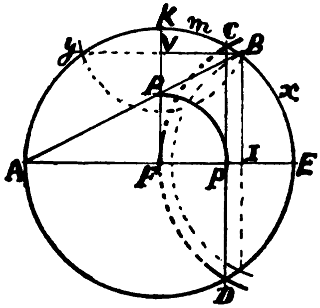


EXHIBIT 15.

28. The squaring chord spans six seventeenths of the circumference. The squaring chord $A\ B$ is obtained by first dividing a given semicircle into two quadrants by a radius as $F\ K$ perpendicular to the diameter $A\ E$. Next, bisect the radius $F\ K$ at P and draw a line through P from A to B, then the squaring chord $A\ B$ is given. This chord is found to span six seventeenths of the circle's circumference.

The arcs evolved from this chord are: the arc $E\ x\ B$, the arc $B\ M\ K$, the arc $K\ y\ A$ and the arc $B\ K\ y\ A$.

The sines evolved are: the versed sines $I\ E$ and $V\ K$, the cosines $I\ F$ and $F\ V$, the right sines $V\ B$ and $I\ B$.

SCHOLIUM.

29. *The squaring chord equals in extent the side of a square equal to the circle in which the chord is drawn. Hence, the squaring chord is the principal factor in squaring the circle as the rectifying chord is the principal factor in squaring the circle's circumference.*

The cube chord, the rectifying chord and the squaring chord, are technically called, the *solving chords*.

POLYGONS.

30. Polygons are figures formed by chords drawn in a circle. There are two kinds of Polygons, namely:

THE REGULAR AND THE DIFFERENTIAL.

Regular polygons have all their sides equal.
Differential polygons have unequal sides.

REGULAR POLYGONS.

31. The first series of the regular polygons comprise:

1. The three sided polygon called the trihedron.
2. The four sided polygon called the square.
3. The six sided polygon called the hexagon.
4. The eight sided polygon called the octagon.
5. The nine sided polygon called the nonagon.

Fig. 1. Fig. 2. Fig. 3.

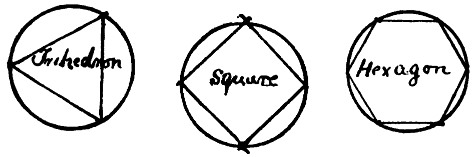


EXHIBIT 16.

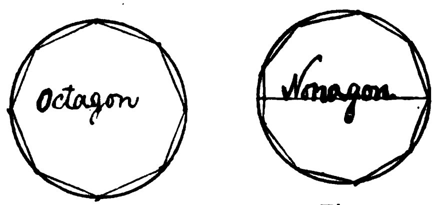


Fig. 4. Fig, 5.

The first series of regular polygons are easily constructed by the processes of bisecting and trisecting as shown by the exhibit 16.

32. The second series of the regular polygons comprise:

1. The five sided polygon called ihe pentagon.
2. The seven sided polygon called the heptagon.
3. The eleven sided polygon.
4. The thirtheen sided polygon.
5. The seventeen sided polygon.
6. The nineteen sided polygon.

When one side of any regular polygon is given, the perimeter of the polygon is also given. Hence, the following six exhibits show only the manner of finding and proving one side of each of the polygons.

CONSTRUCTION.

CHORD OF ONE FIFTH CIRCUMFERENCE.

PENTAGON.

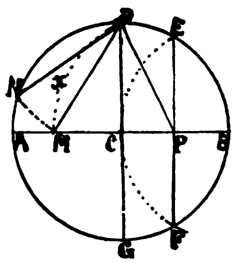


EXHIBIT 17.

33. Quarter the circle by *A B* and *D G*. Bisect radius *C B* at *P* by *E F*. From *P*, as a center, describe with *P D* as radius, the curve *D x M*. From *D*, as a center, describe with *D M*, the curve *M N* which intersects the circumference at *N*. Then draw the line *D N* which is the required chord or side of the pentagon.

CHORD OF ONE SEVENTH CIRCUMFERENCE.

HEPTAGON.

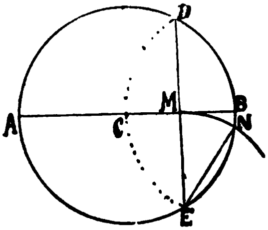


EXHIBIT 18.

34. Bisect the circle by *A B*. Construct the tridrant-chord *D E*. Describe with the right sine *E M*, as a radius, from *E* as a center, the arc *M N* which intersects the circumference at *N*, then draw the line *E N* which is the required chord or one side of a heptagon.

CHORD OF ONE ELEVENTH CIRCUMFERENCE.

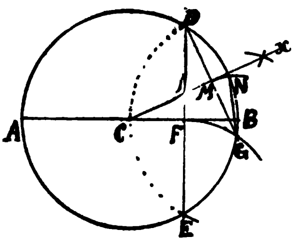


EXHIBIT 19.

35. Bisect the circle by *A B*. Construct the tridrant chord *D E*. Describe with *E F* as radius, from *E* as a center, the curve *F G*. Draw the line *G D*. Bisect *G D* by *x C* and describe with *G M* as radius, from *G* as a center, the curve *M N* which intersects circumference at *N*. Then draw the line *G N* which is the required chord or side of a regular polygon of eleven sides.

CHORD OF ONE THIRTEENTH CIRCUMFERENCE.

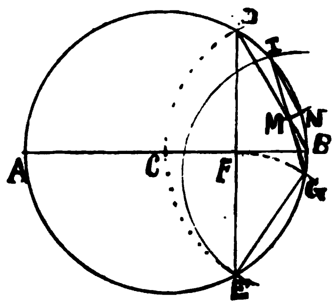


EXHIBIT 20.

36. Bisect the circle by *A B*. Construct the tridrant-chord *D E*. From *E* as a center, describe with *E F* as radius, the curve *F G*. From *G* as a center, with *G E* as a radius, describe the curve, *E x I*, Draw the lines *G I* and *B D*, then the intersecting point *M* is given. From *I* as a center, with *I M* as a radius, describe the curve *M N* which intersects the circumference at *N*. Then draw the line *I N* which is the required chord or one side of a regular polygon of thirteen sides, If properly constructed, *G M* equals one half of *D M* and *B M* equals one half of *I M*.

CHORD OF ONE SEVENTEENTH CIRCUMFERENCE.

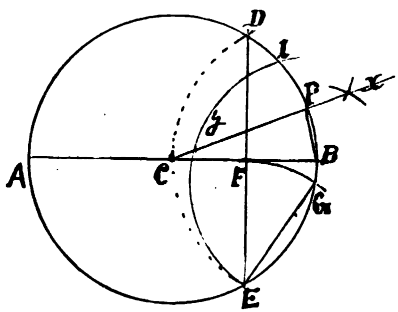


EXHIBIT 21.

37. Bisect the circle by $A\ B$. Construct the tridrant-chord $D\ E$. From E as a center, with $E\ F$ describe the curve $F\ G$. With $G\ E$ for a radius, from G as a center, describe the curve $E\ y\ I$. Bisect the arc $B\ I$ by $C\ x$. Then draw the line $B\ P$ which is the required chord or side of a regular polygon of seventeen sides.

CHORD OF ONE NINETEENTH CIRCUMFERENCE.

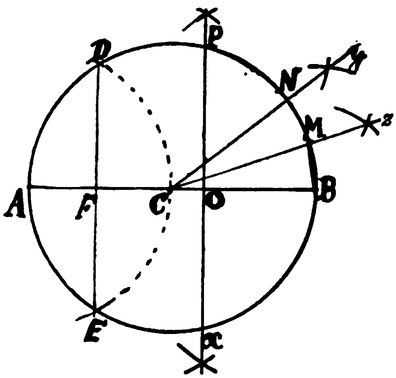


EXHIBIT 22.

38. Bisect the circle and construct the tridrant-chord $D\ E$. Mark the versed sine $A\ F$, bisect $B\ F$ at O with $x\ P$ perpendicular to $A\ B$. Bisect the arc $B\ N\ P$ by $C\ y$ and bisect the arc $B\ M\ N$ by $C\ z$. Then draw the line $B\ M$, which is the required chord or one side of regular polygon of nineteen sides.

SUMMARY OF PART II, SECTION A.

THE THREE TRISECTING FACTORS:

Trisecting Angle—Trisecting Curve—Trisecting Line.

THE THREE KINDS OF COMMENSURATIONAL DIVISION:

Equable—Equitable—Differential.

THE THREE SOLVING CHORDS:

Cube Chord—Rectifying Chord—Squaring Chord.

THE FIRST ELEVEN POLYGONS.

SUGGESTIONS TO THE TUTOR.

To make the beginning of geometric study interesting to the pupil, it is suggested that something should be said about the origin of geometry, as outlined in the preface. It should be explained that the elements presented in the first part of Section A, are derived from nature. It should be explained that the study of part II, consists in finding out, first, by measurement with compasses, and later on, by numerical computation, the true relation of the several geometric elements defined in part I. Careful practice with compasses and ruler is essential to obtain satisfactory results in commensurational division; and as much time and patience should be given to compass practice as generally is given to lessons in penmanship.

SECTION A

PART III

SECTION A.

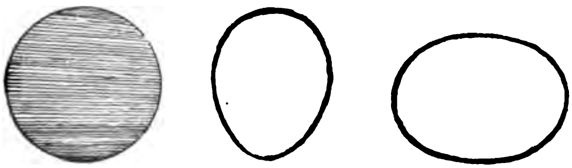
Part III.

BOUNDARIES
OF
GEOMETRIC FORMS.

1. Geometric boundaries are of three distinct kinds, namely: Circumferences, Perimeters and Superfacies.

Circumferences are distinguished by being endless; they emerge from and converge to a common point and consequently exhibit neither points nor angles. Types of these simple boundaries are given in the circle's circumference, in the circumference of the oval and in the circumference of the ellipse.

EXHIBIT 1.

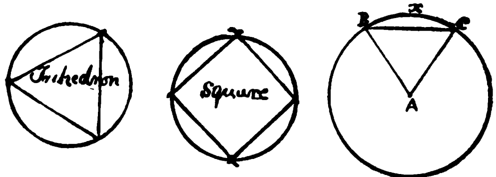


Circle. Oval. Ellipse.

2. Perimeters are composed of lines, points and angles, or, in part of lines and in part of curves. (See the sector in exhibit 2).

Types of perimeters are given in the triangle, in the square and in the sector.

EXHIBIT 2.



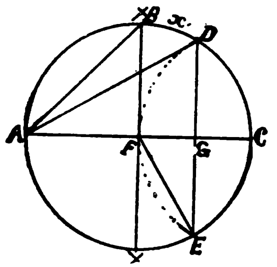
Triangle. Square. Sector.

PLANES.

3. Any blank space, or an even surface, bounded by a circumference or a perimeter, is called a plane. The space contained within a square perimeter is called the square's plane or simply, the square. The space contained within a circles circumference is called the circle's plane or simply, the circle.

As all perimeters are composed of geometric elements, it follows, that every plane and its perimeter occupies some position in and is part of, the circle. (See Axioms, Section A, Part I.)

EXHIBIT 3.



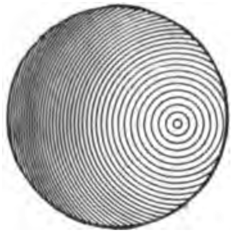
4. The perimeter $A\ B\ F\ A$ forms a plane which is called a prime triangle. Its perimeter is composed of the two radii $F\ A$ and $F\ B$, and of the quadrant-chord $A\ B$. The position occupied by that plane in the circle is the position of a quadrant's angle plane, or, the plane of a quadrant minus the segment. The plane $A\ D\ G\ A$ is bounded by the trident-chord $A\ D$, by the right sine $D\ G$, and by the diameter $A\ C$ minus the versed sine $C\ G$. The plane $B\ F\ E\ D\ x\ B$ is bounded by the radii $F\ B$ and $F\ E$, by the tridrant-chord $D\ E$ and by the arc $D\ x\ B$.

VOLUMES.

5. A space bounded by planes (flat or curved) is called a volume. The planes which compose a volume's boundary are called superfacies. A volume represented by bulk is called a solid.

THE GLOBE.

6. The simplest geometric solid is bounded by one uniform convex plane called the globe's face. The solid under this face is called the globe.

EXHIBIT 4.  (The Globe)

THE SPHERE.

7. The space occupied by a globe is called a sphere. The boundaries of a sphere is described as a *uniform concavity* obtained by revolving a circle on its diameter as an axis.*

SCHOLIUM

8. Given superfacies in a globe form contain a *greater* volume or bulk than the same superfacies cast into any other form.

A given distance formed into a circular boundary contains a *greater* plane than the same circumference cast into any other form.

A square perimeter contains a *greater* plane than the same perimeter cast into any other *angular* form.

GENERAL CLASSIFICATION OF PLANES.

9. Circles — Segments — Sectors — Triangles—Rectangles—Rhombs—Trapezes—Ovals—Ellipses.

DESCRIPTION.

The first three forms (circles, segments and sectors) have already been described and defined in the exhibits 6, 7, 8, paragraphs 14, 15, 16, in Part I, Section A.

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* Phenomenally, the mutual relation of the globe and the sphere is represented by a soapbubble blown through a pipestem. That is, the inblown breath of air which expands the bubble represents the capacity of the sphere; the concave superfacies of the soapy film which bounds the inblown air, represents the sphere's boundary; while the exterior convex superfacies of the film marks the face of a globe.

TRIANGLES.

10. All triangles have three sides, three angles and three points of unity.

11. Triangles subdivide into prime triangles, trihedrons, right-angled triangles, isosceles triangles, scalene triangles.

THE PRIME TRIANGLE.

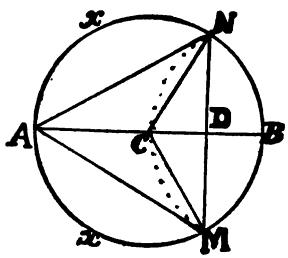
12. The prime triangle has two sides equal, each of which, is technically called a *leg*. The third side, which is the longest line, is technically called the *hypotenuse*. Every prime triangle has one right-angle and two acute angles. Hence, the prime triangle is a right-angled triangle but differs from every other of the class in having two sides equal and two equal angles. Exhibit 3, paragraph 4, shows the form of a prime triangle.

Two prime triangles forms one square.

The geometric elements which compose the perimeter of the prime triangle, are either radii and chord, diameter and chords, or sines and chord.

THE TRIHEDRON.

EXHIBIT 5.



13. The trihedron has all its sides equal and all its angles acute. Its perimeter is composed of tridrant-chords.

Exhibit 5, shows the elements of the trihedron's perimeter. The point A represents its vertex, the line $M\ N$ represents its base, the line $A\ D$ represents its altitude.

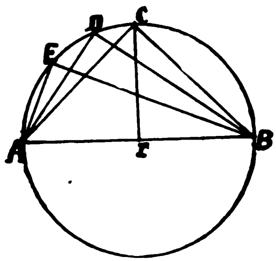
SCHOLIUM.

14. The terms: base, altitude and vertex, have fixed relations to planes. In right-angled triangles and in scalene triangles the longest line is the base. In isosceles triangles the odd side is the base. The vertex of a triangle is the angle point opposite the base. The altitude of a triangle is the perpendicular distance from vertex to base.

RIGHT-ANGLED TRIANGLES.

15. All right-angled triangles are contained in the semicircle and any two chords drawn in a semicircle, which chords form a right angle also form a right-angled triangle on the diameter of the circle.

EXHIBIT 6.



Thus: $A\ B\ C$, $A\ B\ D$, $A\ B\ E$, are all right-angled triangles having the diameter $A\ B$ for a common hypotenuse and the several chords for legs.

THE CUBE TRIANGLE.

16. The cube triangle is a right-angled triangle contained in every cube. Its hypotenuse represents the longest line in a cube (diagonal of volume), its minor leg represents the edge of the cube, and its major leg represents the diagonal of the cube's face.

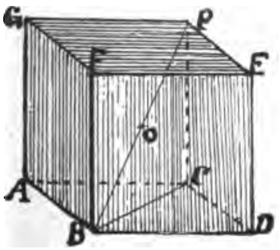


EXHIBIT 7.

Exhibit 7, shows the cube triangle $B\ C\ P\ B$. $P\ C$ represents the edge of a cube; $C\ B$ represents the diagonal of the cube's face; $P\ O\ B$ represents the diagonal of the cube's volume.

ISOSCELES TRIANGLES.

17. Isosceles triangles have no right angles. Two sides are equal, the third is odd, or not equal to the other two The two equal sides are called legs, the odd side is called the base.

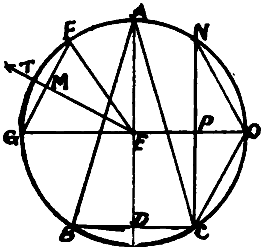


EXHIBIT 8.

18. All isosceles triangles are contained in sectors, and each one, is the angle plane of a sector.

$A\ B\ C$ is an acute angled isosceles triangle composed in its perimeter of three chords, $A\ B$, $B\ C$, $C\ A$. Its base is $B\ C$, its vertex is A, its altitude is $A\ D$.

$C\ N\ O$ is an obtuse angled isosceles triangle composed in its perimeter of the three chords: $N\ C$, $N\ O$, $O\ C$. Its altitude is $P\ O$.

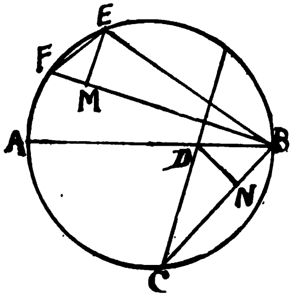
$G\ E\ F$ is an acute isosceles triangle composed in its perimeter of two radii and one chord as shown by $G\ E$, $E\ F$, $F\ G$. Its altitude is $M\ E$.

SCALENE TRIANGLES.

19. Scalene triangles have three unequal sides and no right angle.

All scalene triangles contain two right-angled triangles which are produced by the altitude represented by a line drawn from the vertex perpendicular to the base. In this class of triangles the longest side represents the base.

EXHIBIT 9.



The triangles $B\ F\ E$ and $B\ D\ C$ are scalene triangles, the respective altitudes of which, are: $E\,M$ and $D\,N$.

All scalene triangles are contained in segments less than the semicircle.

RECTANGLES.

20. Rectangles are four sided planes which have four right angles and opposite sides parallel. The simplest of the rectangles is called the square, which has already been described in Sect A, Part II, paragraph 28, fig. 2 as one of the polygons.

THE SQUARE.

21. The square differs from any other rectangle, in this, that all its sides are equal, whereas, every other rectangle has two major sides and two minor sides.

Fig. 1. Fig. 2.

EXHIBIT 10.

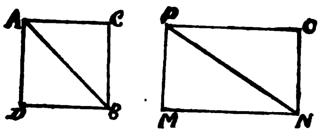


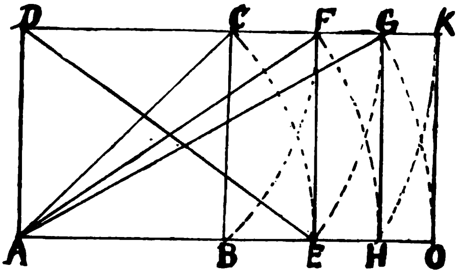
Fig. 1, represents the square; Fig. 2, represents the rectangle. The longest distance from point to point in any rectangle is called its diagonal. The lines $A\ B$ and $P\ N$ represent the respective diagonals of the square $A\ B\ C\ D$ and of the rectangle $M\ O\ N\ P$.

THE PRINCIPAL RECTANGLES ARE.

22. The *prime* rectangle—the *cube* rectangle— the *prime square root.*

Exhibit 11, represents the prime rectangle *A F E D*, the cube rectangle *A G H D*, and the prime square root *A K O D*. The side *A E* equals the diagonal *A C*, the side *A H* equals the diagonal *A F*, the side *A O* equals the diagonal *A G*.

EXHIBIT 11.



THE PRIME RECTANGLE.

23. The perimeter of the prime rectangle equals two sides of the square *A C D B* and two diagonals of the same square. That is: the minor sides *A D* and *E F* are proportioned to the major sides *A E* and *D F* as are sides of squares to diagonals of squares. The plane of the prime rectangle equals two cube triangles as shown by the figures *A D E A* and *E F D E*.

THE CUBE RECTANGLE.

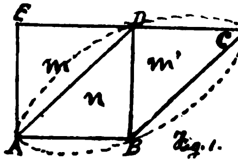
24. The perimeter of the cube rectangle is composed of two minor sides: *A D* and *H G*; and of two major sides: *A H* and *D G*. The major sides are each equal to the diagonal *A F* of the prime rectangle *A F E D*.

THE PRIME SQUARE ROOT.

25. The perimeter of the prime square root is composed of two diagonals of the cube rectangle and two sides of the square *A C D B*. Hence, the plane of the prime square root equals two squares as *A C B D*, and it follows, that the perimeter equals six sides of the square *A C D B*.

RHOMBS.

26. Rhombs are four sided planes the opposite sides of which are parallel, while the angles are acute and obtuse. These planes can be composed of all kinds of triangles as shown in Exhibit 12.



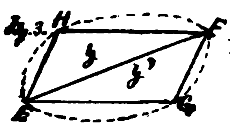


Fig. 2

Fig 3

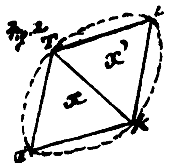


EXHIBIT 12,

All rhombs have a minor and a major diagonal and are circumscribed by elliptic curves.

Exhibit 12, shows by figure 1, a rhomb composed of two prime triangles (*n* and *m*) bounded by the perimeter *A C D B*. Fig. 2 shows the perimeter (*E G H F*) of a rhomb composed of two scalene triangles (*y* and *y*'). Fig. 3 shows the perimeter (*I L K T*) of a rhomb composed of two trihedrons (*x* and *x*').

27. A rhomb which has all its sides equal is called a rhombus. All other rhombs are called rhomboids. In exhibit 12, the figures 1 and 2 represent rhomboids, figure 3 represents the rhombus.

THE PRIME RHOMBUS.

28. The prime rhombus has a minor diagonal equal to two radii of a given circle, a major diagonal equal to two quadrant chords, and a perimeter which equals four tridrant chords, all, of the same given circle. Hence, the plane of the prime rhombus is composed of four right-angled triangles, each of which is a cube triangle.

Fig. 1

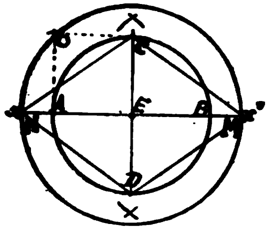


Fig. 2

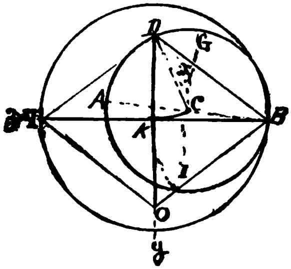


EXHIBIT 13.

29. Fig. 1 shows a circle quartered by the diameters *A B* and *C D*. On the radius *E A* construct the square *A O C E*, extend the diameter *A B* indefinitely to *x* and *x'*. Describe with the diagonal *E O* the circle over the diameter *M N*, then it is found and constructively shown that *C D* is proportioned to *N M* as the side of a square is to the diagonal of the square. And it follows, when the lines *C M*, *M D*, *D N* and *N C* are drawn, the perimeter of a prime rhombus is constructed according to definition.

30. Fig. 2, shows the prime rhombus constructed in a different manner, so as to show that the elements which compose the perimeters of the four cube-triangles composing the plane of the rhombus, are proportioned as are radius, quadrant-chord and tridrant-chord in any given circle. Thus, *A C B* is the diameter of a given circle. Quarter that circle by the points *A*, *G*, *B*, *I*. Draw the tridrant-chord *B D*. Draw the quadrant-chord *B I*. With radius *D C*, from *D* as a center, describe the arc *C K*, and with the quadrant-chord *B I*, as a radius, from *B* as a center, describe the arc *I K*, which two arcs, give the intersecting point *K* which is found to be the center of the greather circle described over the diameter *B T*. Now, draw a line from *D* through *K* to *y*, and lay off on *D y* the *line* *K D* at *O*.

Finally, draw lines from D to T, from T to O, from O to B and from B to D, then, the perimeter of a prime rhombus is given by the figure $D\ T\ O\ B\ D$, and the plane of the figure is shown to be composed of four cube-triangles as defined.

TRAPEZES.

31. Trapezes are of three kinds: *Irregular* Trapezes—*Trapeziums*—*Trapezoids*.

Irregular trapezes are represented by four-sided figures, all four sides of which, are unequal and no two sides are parallel.

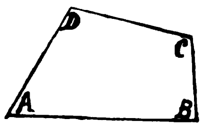


EXHIBIT 14.

$A\ B\ C\ D$ represents an irregular trapeze.

TRAPEZIUMS.

32. Trapeziums are four sided planes which have no right angles, two parallel sides, two equal acute angles and two equal obtuse angles.

THE PRIME TRAPEZIUM.

33. The parallel sides of the *prime trapezium* are proportioned as side and diagonal of squares, or, as radius of circle is to the quadrant-chord; each of the major sides is proportioned to the least side as a tridrant-chord is to radius; and each of the longest sides is proportioned to the side parallel to the least as a tridrant-chord is to the quadrant-chord.

Manner of Constructing the Prime Trapezium.

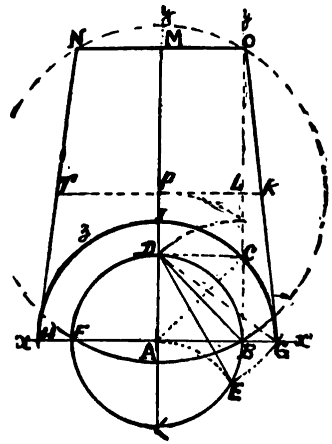


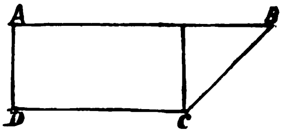
EXHIBIT 15.

With any radius, as for example *A B*, construct a circle. Quarter that circle and construct on its radius a square as *A C B D*. Extend the diameter *F B* indefinitely to *x* and *x'*. In the circle construct a tridrant-chord as *D E* and a quadrant-chord as *D B* (which chord equals the diagonal *A C*). Now, lay off on *A x'* the diagonal *A C* at *G*, then the perimeter of a cube triangle is formed and represented by the figure *A D G A*. Next, proceed to describe with the radius *A G*, from *A* as a center, a semi-circle as represented by the figure *H z I C G A H*. Extend the line *A D* indefinitely to *y*; extend the line *B C* to *y'*; and construct on *A y* and *B y'* the cube rectangle *P B L A*. From *P* as a center, with the distance *P B* as a radius, construct a circle which intersects the line *B y'* at *O*. Now draw from *O*, the line *O G*, then the longest side of the prime trapezium is given, and the bisecting point *K* of that side is found to be a lineal extension of *P L*. Complete the figure by drawing the chord *O N* parallel to *A B*; draw the line *N H*; extend *A P* to *M* and *K P* to *T*; then the prime trapezium *H G O N H* is constructed according to definition.

TRAPEZOIDS.

34. Trapezoids have two sides of their perimeters parallel, two right angles, one acute and one obtuse angle. Hence, all trapezoids are composed in their planes of one square, or a rectangle, and one prime triangle.

EXHIBIT 16.



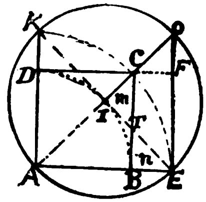
A B C D A represents a trapezoid.

THE PRINCIPAL TRAPEZOIDS.

35. The principal trapezoids are: the prime trapezoid and the tri-trapezoid.

Manner of Constructing the Prime Trapezoid.

EXHIBIT 17.



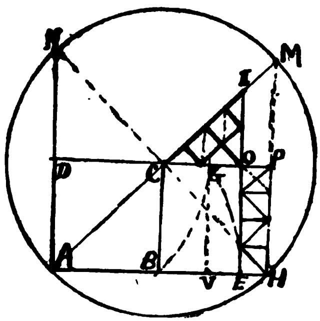
A B C D is a given square. Develop from this square the prime rectangle *A F E D*. Extend the lines *E F* and *A C* until they meet at the point *O*, then, a prime trapezoid is described by the figure *B C O E B*.

36. The prime trapezoid in exhibit 17, is shown to equal in area, one-half the plane of a square constructed on the least of its parallel sides, that is, the side *B C*. For if the line *A O* is bisected at *I*, and a circle is described from *I*, as a center, with *I A* for a radius, it is found, that a line passing through the center *I*, from *E* to *K*, produces the two equal triangles *m* and *n*. Now, it is self-evident, that the triangle *I O E* equals

one-half the square *A C B D*, or the triangle *C B A*, because, if the part *m*, which equals the part *n*, is added to the figure *T C O E T*, the same area is contained in the trapezoid as in one-half the given square.

Manner of Constructing the Tri-Trapezoid.

EXHIBIT 18.



37. The tri-trapezoid equals in its plane the area of a square constructed on the least of the trapezoid's parallel sides.

A C B D is a given square. With *C A*, as radius, from *C*, as a center, describe a circle and extend the radius *A C* to *M*. Quarter the circle by the line *H N*. Now construct, from the square *A C B D*, the prime rectangle *A G V D*, and the cube-rectangle *A O E D*, and the prime square root *A P H D*. Extend the line *E O* to *I*, then a tri-trapezoid is given by the figure *B E I C B*. For it is found — if the rectangle *E P O H* is divided into 8 equal parts in triangle form, these 8 prime triangles equal the one prime triangle *C O I C*. Now, since the square *B P C H* equals the square *A C B D*, it follows, that the rectangle *B O C E* and the prime triangle *C I O C*, which together form the tri-trapezoid *B I E C*, also equal the square (*A C B D*) constructed on the least of the parallel sides of the trapezoid.

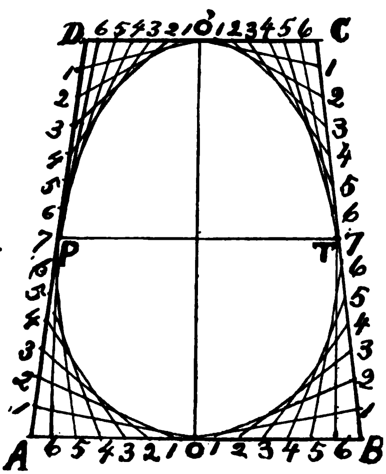
OVALS.

38. Ovals are planes bounded by endless curves inscribed in trapeziums.

THE PERFECT OVAL.

The perfect oval is constructed as the following exhibit shows.

EXHIBIT 19.



39. Construct the prime trapezium *A C D B.* * Divide commeasurately ** the several sides of the trapezium's perimeter into fourteen equal parts, and number these respectively as shown in the exhibit, from *o* to *1*, from *1* to *2*, from *2* to *3*, from *3* to *4*, from *4* to *5*, from *5* to *6* and from *6* to *7*. Then a series of points are given which shape the curve of the perfect oval.

ELLIPSES.

40. Ellipses are planes bounded by endless curves inscribed in rectangles.

---

* See exhibit 15.

** See commensurational divisions, Section A, Part 2, § 12.

[illegible] diameter

[illegible] Bisect A
[illegible] perpendiculars as A [illegible]
[illegible] F, and
[illegible] and
[illegible] parallel
[illegible] the line AB
[illegible] sides and
[illegible] the perimeter
[illegible] the commen-
[illegible] previously in
[illegible] series of points
[illegible] perfect ellipse.

The perfect ellipse circumscribes a prime rhombus, for when lines are drawn from E to I, from I to C, from C to K and from K to E, then the perimeter of a rhombus is given, the diameters of which are proportioned as sides and diagonals of squares, and it follows, that four cube triangles are also given. (See definitions of cube triangles and prime rhombuses exhibit 13.)

GENERAL CLASSIFICATION OF VOLUMES.

42. THE SPHERE AND THE GLOBE. — THE CONE. — THE CYLINDER. — THE CUBE. — THE PARALLOG. — THE PYRAMID. — THE TETRAHEDRON.

These volumes must be presented by solids which of themselves explain the several dimensions.

DIMENSIONS OF THE VOLUMES.

43. There are six dimensions in geometric volumes, namely:

DEPTH, — BASE, — ALTITUDE, — LENGTH, — BREADTH, — SLANTS.

These several dimensions are exhibited by the socalled vertical sections obtained by cutting the volumes or solids through in a manner so as to expose to view the greatest plane.

VERTICAL SECTION OF THE GLOBE.

44. When a globe is bisected, the greatest plane of the bulk is given in the form of a circle, the center of which represents the sphere's center and the circumference of which represents the sphere's periphery. The radius of the circle represents the sphere's depth, the diameter of the circle represents the sphere's altitude, its length and its breadth; the plane of the circle represents the sphere's base.

SUPERFACIES OF THE GLOBE.

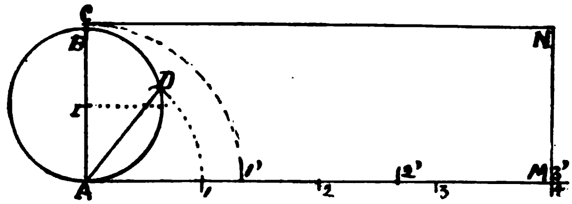


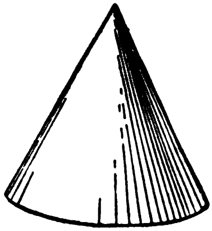
EXHIBIT 21.

45. The superfacies of the globe equal four times the base, that is, the greatest circle contained in the globe. Exhibit 21 shows the superfacies in a rectangle plane obtained by the following process:

Let *I A* and *I B* represent radii of the great circle. Let *A D* represent a distance equal to one-fourth of the circle's circumference, then four *A D* equal the globe's periphery. In the diagram, the line *A M* represents a distance equal to the periphery, rectified into one straight line. The figures *1*, *2*, *3*, *4*, represent four *A D*. Now, if this same line is trisected into three equal parts as shown by the figures *1'*, *2'*, *3'*, and with the distance *A I'*, an arc is described from *A* as a center, a distance greater than the diameter *A B* is obtained and shown by the line *A C*. From *C*, construct the line *C N* parallel to *A M* and of equal length to *A M*, and join the points *M* and *N* by the line *M N*, then, the rectangle *A N C M* equals in area four times the given circle-plane. Such a rectangle represents a space equal to the superfacies of a sphere or globe of which the circle over the diameter *A B* is the base, or the great circle. Hence it is shown, that a rectangle composed in perimeter of two-tridrant arcs and of eight quadrant arcs equal the superfacies of a given globe.

DIMENSIONS OF THE CONE.

FIG. I.



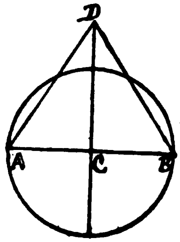


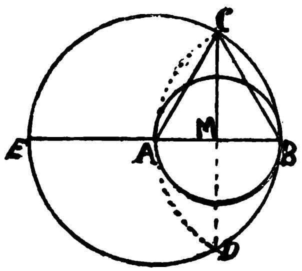
FIG. 2.

EXHIBIT 22.

46. Fig. 1 shows the cone volume; Fig. 2 shows the vertical section of the cone by the figure *A D B A*. The base of the cone is shown by the circle over the diameter *A C B*. The altitude of the cone is shown by the line *C D*. The slants are shown by the lines *A D* and *B D*.

SUPERFACIES OF THE CONE.

FIG. 1.



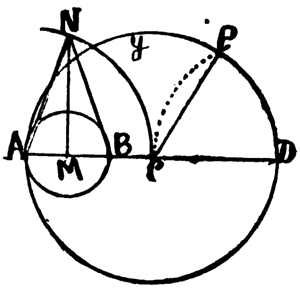


FIG. 2.

EXHIBIT 23.

47. The circle over the given radius *A B* (fig. 1) represents the base of a given cone, the vertical section of which is represented by the triangle *A B C*. The altitude of the given cone is shown, by the line *M C*, to be the right sine of a sextant in the circle of which *A B* is radius; and it is also shown that the slants (*A C* and *C B*) equal, each, the radius of the greater circle over the diameter *E B*. Now, it is found, when the circumference of the given cone's base is rectified (or straightened out) and compared to the greater circle's rectified circumfer-

ence, that one-half the greater circle's circumference, equals the whole of the lesser circle's circumference. Hence, the curved superfacies of the given cone is found in the section *B C E A B*. This may be mechanically demonstrated by cutting out the section and forming it into a funnel-form.

48. Likewise: the circle over *A B* (fig. 2) represents the base of another cone, the vertical section of which is represented by the triangle *A B N A*. The altitude of the cone is given by the line *M N*, and the two slants, *A N* and *N B*, are each found to equal the radius *C A* of the greater circle. By measurement it is found that the diameter *A B* of the cone's base is one-third of the diameter (*A D*) of the circle constructed with a radius equal to the slant of the cone. Finally it is found, that one-third of the greater circle's circumference equals the circumference of the cone's base. This can be mechanically shown by cutting out the sector *A C P y A* and form it into a funnel shape.

SCHOLIUM 1.

49. The above demonstrations show that certain fixed relations exist between diameter of a cone's base and the circumference of a circle constructed with a radius equal to the slant of the given cone.

The first demonstration shows: when diameter of a cone's base is one-half a given circle's diameter and when the cone's slant equals radius, then the section's arc shall equal one-half the given circle's circumference.

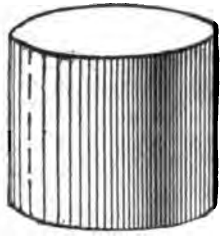
The second demonstration shows: when a tridrant is given as the curved part of a cone's superfacies, then the diameter of the cone's base shall equal one-third the given circle's diameter; and so on, the arc of the sector or section required for the curved superfacies of any given cone, shall be proportioned to the circumference as the diameter of the cone's base is to the diameter of the given circle.

SCHOLIUM 2.

50. Solid cones cut diagonally, produce elliptic planes. *Spheric* cones cut diagonally and vertically produce Parabolas, Hyperbolas and other oval forms. (For explanation of these conic sections see supplement.)

VERTICAL SECTION OF THE CYLINDER.

FIG. 1.



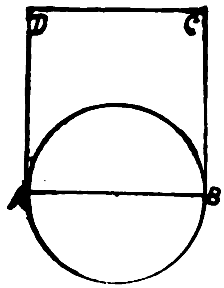


FIG. 2.

EXHIBIT 24.

51. Fig. 1 shows the solid; Fig. 2 shows the vertical section. The base of the cylinder is represented by the circle over the diamemeter *A B*. The altitude of the cylinder is represented by the lines *A D* or *C B*, and the greatest plane of the cylinder's volume, is shown by the rectangle *A C D B*.

SUPERFACIES OF THE CYLINDER.

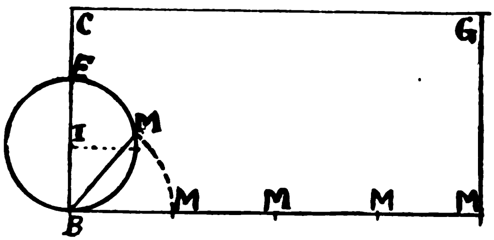


EXHIBIT 25.

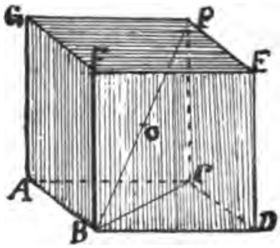
52. Describe the circular base over the cylinder's diameter *B E*. Draw the rectifying chord (*B M*) equal to one-fourth the circumference of the base. Mark the cylinder's altitude by the line *B C*. Lay off on a line perpendicular to *C B*, from *B*, four *B M*, then, a minor and a major side of the rectangle *B G C M* are given, and when this rectangular plane is formed into a tube, it is found to equal in area the curved superfacies of the given cylinder. When the area of the two bases of the cylinder is added to the rectangle the whole of the bounding superfacies of the cylinder is given.

SCHOLIUM.

53. A solid cylinder cut diagonally at various angles produces the several kind of ellipses, called the cylindrical sections.

VERTICAL SECTION OF THE CUBE.

EXHIBIT 26.

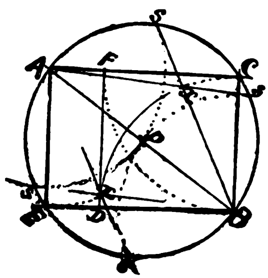


THE SOLID CUBE.

54. The Cube is a volume contained under six equal square planes, five of which are called "faces", while the sixth, on which the cube rests, is called the cube's base. The lineal side of the cube's base is called the cube's edge and must not be mistaken for the root of the cube, which is a parrallog and a part of the volume. *A B* or *B D* represent the edge of the cube; the line *B C* represents the diagonal of the cube's *face*; the longest line in any cube is called the diagonal of the cube, which, is represented by the line *B O P*. The point *O* which evenly divides the longest line marks the center of the cube, and like the center of the globe, the same point marks the limit to the volume's depth.

THE VERTICAL SECTION.

EXHIBIT 27.

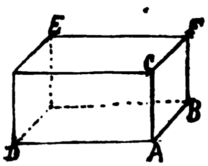


55. The vertical section of a cube is a rectangle composed of two cube-triangles having the cube's diagonal for a common hypotenuse.

The exhibit 27 shows the two cube triangles, *A B C A* and *A B E A*, constructed within a circle, the diameter of which is the common hypotenuse. Thus, the vertical section of the cube is shown by the rectangle *A B C E*; and it is found, by constructing the square *C D B F*, that the vertical section equals the prime rectangle as defined by exhibit 11, §22.

THE PARALLOG.

EXHIBIT 28.



56. The parallog is bounded by six rectangles which form the volume's superfacies, or by two squares and four rectangles. The opposite faces are parallel and the adjacent faces are perpendicular to each other.

The length of the parallog is greater than its altitude or its breadth, consequently, four of its faces are major and two of its faces are minor. When the two minor faces are squares, the parallog is said to be prime.

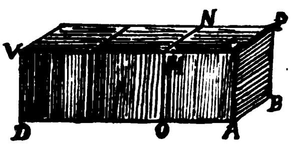
GEOMETRIC ROOTS.

57. All prime parallogs divided into a number of equal parts in cube form and fractions of a cube, constitute cube roots.

Square roots are rectangles formed by the major face of a parallog divided into a number of equal parts in square form and fractions of a square.

INTEGRAL ROOTS AND SURD ROOTS.

FIG. 1.



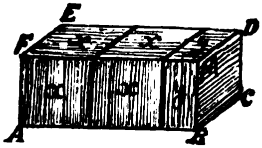


FIG. 2.

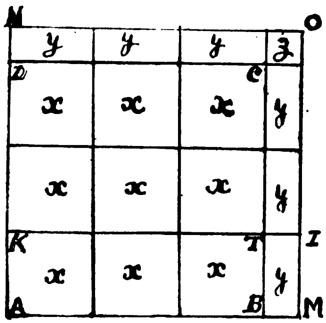
EXHIBIT 29.

58. The parallog $D\ A\ B\ P\ V\ D$ (fig. 1) represents an integral cube root, composed of three equal unit cubes, as the one marked $A\ B\ P\ N\ M\ O\ A$. The root is called integral because the unit measure is an aliquot divisor for the parallog which is a part of a cube's volume. The cube raised from any integral geometric root is also called an integral cube, because, the same cube unit which divides the root, without a remainder, also divides the whole cube without a remainder.

59. Fig. 2 represents a cube surd root. The unit measure is the part x in cube form. The exhibit shows, that the parallog $A\ B\ C\ D\ E\ F\ A$ is composed of two units and the fractional part of a unit marked y. Now, it is evident, that the given unit x is an aliquant divisor for the given root, and so is also the same unit when used as a divisor for the cube volume raised on the same surd root.

INTEGRAL ROOTS AND SURD ROOTS OF SQUARES.

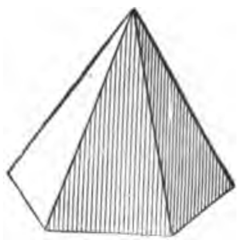
EXHIBIT 30.



60. *A B T K A* represents the plane of a given integral square root, the unit of which is shown by the square unit *x*. The square *A C D B* is constructed and raised on that root, and it follows, that the square is integral since the given unit *x* is an aliquot divisor for the whole square which it divides into nine equal parts. *A I M K* is the plane of a surd square root, and when a square is constructed or raised on that root, a surd square results as shown by the square *A O M N* which is composed of nine whole units and a number of fractions, marked *y*.

THE PYRAMID.

EXHIBIT 31.

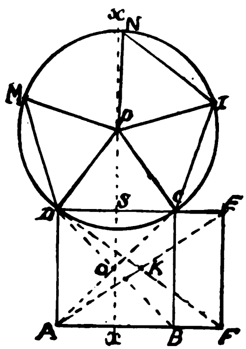


61. Pyramids are angular cones composed, in superfacies, of slanting triangles and a base in the form of a polygon or a square.

The most important of the pyramids is called the prime pyramid, six of which compose a cube volume.

THE PRIME PYRAMID.

EXHIBIT 32.



61. This pyramid's base is a square, representing the face of a cube. The pyramid's four triangular faces equal, each, an isosceles triangle, composed in perimeter of the cube's edge and the cube's diagonal, while the altitude of the triangle's plane equals one-half the diameter of the cube's face. The altitude of the pyramid equals one-half the cube's edge. The major slant equals one-half the cube's diagonal. The minor slant equals one-half the facial diagonal of the cube.

SUPERFACIES OF THE PRIME PYRAMID.

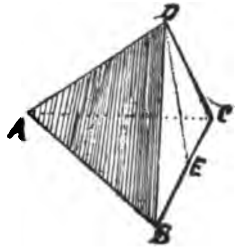
62. Construct the prime rectangle $A\ E\ D\ F$, which gives the cube's face $A\ C\ B\ D$. Draw the diagonal $E\ A$ and $F\ D$, which give the center K. Draw the diagonals $D\ B$ and $A\ C$, which give the center O. Then it is shown, that $D\ O$ is one-half of the facial diagonal, and $D\ K$ is one-half of the cube's diagonal.

Bisect the square $A\ B\ C\ D$ by the line $x\ x'$ and mark with $D\ K$, as a radius, from D as a center, the point P, which is equidistant from D and C. Then draw the lines $P\ D$ and $P\ C$, which give the triangle $D\ P\ C$. The altitude ($P\ S$) of this triangle is found to be equal to the distance $D\ O$. Thus, it is shown, that $D\ C$ equals the edge of the cube; that $D\ P$ and $P\ C$ equal, each, one-half the cube's diagonal $D\ K$; and finally it is shown, that $P\ S$ equals one-half the diagonal of the cube's face, represented by $D\ O$.

Now, if from the center P, with the radius $P\ D$, a circle is described and four chords as $M\ D$, $D\ C$, $C\ I$ and $I\ N$ are drawn in that circle, then, four sectors are formed as $M\ P\ D$, $P\ D\ C$, $C\ P\ I$ and $I\ P\ N$. Cut these four sectors out in one secction, as the section $N\ P\ M\ D\ C\ I\ N$ and join together the slants $P\ M$ and $P\ N$, so that the several sectors meet and form the apex P. Then the slanting superfacies of the prime pyramid are given, and it is found that the perimeter thus formed by the four triangles, fit the perimeter of the given cube's face. Six such pyramids equal the volume of the cube.

THE TETRAHEDRON.

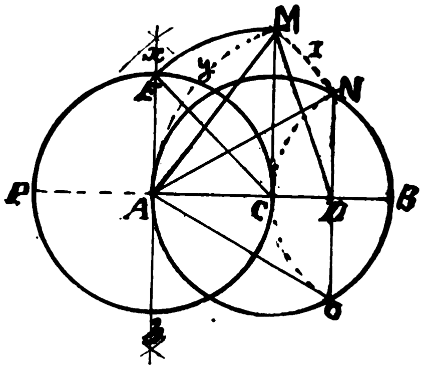
EXHIBIT 33.



63. The tetrahedron's volume is bounded by four trihedrons, one of which is called the base and on this the volume rests. The other three triangles are called faces. *A B C A* represents the base. *D B* represents the major slant. *E D* represents the minor slant; and a line from apex to center of base represents the altitude of the tetrahedron.

THE VERTICAL SECTION.

EXHIBIT 34.



64. The vertical section of a tetrahedron is an isosceles triangle composed of two rightangled triangles as shown in the exhibit by the figure *A D M A*.

One of the two component triangles (*A C M*) is a cube triangle; the other (*C D M*) is proportioned in the several parts of its perimeter, as are the quadrant chord, the co-sine of a sextant, and three-fourths of the diameter, in any circle.

65. The vertical section of a tetrahedron is obtained as follows:

Inscribe a trihedron in a circle, as the trihedron *A N O* in the circle over the diameter *A B* of which *C* is the center.

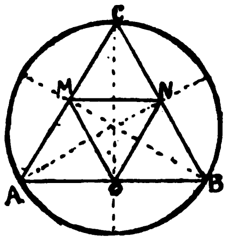
Construct on the center C, a perpendicular line parallel to $N\ O$, as the line $C\ M$ and extend indefinitely. With the tridrant chord $A\ N$, from A as a center, describe the arc $N\ I\ M$ which intersects the indefinite line at M. With $D\ A$ as a radius, from D as a center, describe the arc $A\ y\ M$ which intersects at M. Then, the vertical section is given by the figure $A\ D\ M\ A$, and the base of the same tetrahedron is given by the figure $A\ N\ O\ A$.

66. Now, if another circle of equal diameter is constructed by the radius $A\ C$, from A as a center, then it can be shown that the altitude of the given tetrahedron equals the quadrant chord of the circle in which the base of the tetrahedron is inscribed. For, let $C\ E$ represent the quadrant chord, and with it, as a radius, describe the arc $E\ M$, from the center C, then, because that arc intersects at the apex point M, it is shown, that the altitude of the tetrahedron equals the quadrant chord of the circle in which the base is inscribed.

SUPERFACIES OF THE TETRAHEDRON.

67. The superfacies of any tetrahedron are obtained from and comprehended in an inscribed trihedron, as exhibit 35 shows.

EXHIBIT 35.



Bisect the three sides of the given inscribed trihedron $A\ B\ C$, and mark the bisecting points $M\ N\ O$. In the first given trihedron inscribe an other as the figure $O\ M\ N$. Make this minor trihedron the base of the required tetrahedron, and after cutting out the major trihedron, fold the three triangles $A\ O\ M$, $M\ N\ C$, $O\ N\ B$ so they form the apex out of the points A, B, C, then, the superfacies of a tetrahedron are given and shown to be equal to the triangle $A\ B\ C$.

SUMMARY OF PART III, SECTION A.

THE THREE KINDS OF BOUNDARIES:
Circumferences—Perimeters—Superfacies.

THE NINE KINDS OF PLANES:
Circles—Segments—Sectors — Triangles — Rectangles — Rhombs—Trapezes—Ovals—Ellipses.

THE SEVEN KINDS OF VOLUMES:
Spheres and Globes—Cones—Cylinders — Cubes — Parallogs—Pyramids—Tetrahedrons.

THE SEVEN KINDS OF GEOMETRIC ROOTS:
Linear roots — Roots of planes — Roots of volumes— Square roots — Cube roots — Integral roots — Surd roots.

VERTICAL SECTIONS—CONIC SECTIONS—CYLINDRICAL SECTIONS.

SUPPLEMENT.

TUTORS' SCHOLIUM.

(SECTION A, PART III.)

In the Classification of geometric forms, several new terms are introduced to express new aspects of geometric quantities, as for example, the term *parallog*, which takes the place of "parallelopiped." In some cases, new definitions are given to former technical terms which have not hitherto been properly defined, as for example, the given definition of geometric roots. Particular pains should be taken by the tutor to impress on the pupil's mind a distinct difference in the meaning of *geometric* squares and cubes, and the socalled second and third power of abstract numbers, represented by figures like these :

2^2, 3^3 etc. Geometric roots express *form* as well as magnitude, and geometric roots are either a portion of some given line, a part of some given plane, or a part of some given volume, all of which can be properly represented by mechanical devices. The *Spheric* cone is another innovation represented by a truncated solid cone, capped with a fitting hemisphere, and resting on another hemisphere fitting the cone's base. This device explains itself when seen, and its use is of the highest importance in determining the exact forms of the *Parobola* and the *Hyperbola.* These two last named forms are not given a place under "general classification" as they are subdivisions of the oval order and have not hitherto been considered definable with any terminal boundaries. They are better classed under physics. The vertical sections and their uses present new fields for inquiry and it is suggested that particular attention be given to that branch of the study. The methods described, by which bounding superfacies are obtained in plane figures, will greatly help the pupil to remember the several dimensions of given volumes. The study can be made still more interesting, by letting the pupil draw the figure representing the superfacies, have him cut out the figure, and make him fold the paper so as to shape the boundary of the volume.

Part II and III of Section A may be considered the book of geometric object-teaching and ought to be of particular interest to both tutor and pupil, since it presents all the fundamental principles of pure geometry, in a simple and attractive manner.

(Conwing)

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SECTION B

PART I

PREFACE.

(TO THE TUTOR.)

ORIGIN OF NUMBERS.

Numbers originated with the idea of counting single items and thinking over how often the same thought, or thing, occurred to one's mind. Primitive man may have counted his fingers and toes, or he may have tried to count the stars at night. He noticed the periodical rising and setting of the sun, and may be, he counted "one" each time he saw that shining orb come and go. Then came to him the idea of keeping account by noting the number of times the same or similar events occurred. This notion suggested the use of representative marks, which later resulted in devising numerical systems of computing figures. Whatever kind of figures were used prior to historical record is not in our province to speculate upon, but this we know, that every numerical system must have three distinct principles represented in its foundation, namely: *singularity*, *plurality*, *totality*. The first of these principles relates to things that are, or to things which by common agreement shall be — indivisible. Whatever is indivisible, either real or by agreement, must necessarily be a fraction or part of something else, since everything whole can be divided into parts.

This indivisible fraction, expressing the principle of absolute singularity is symbolically represented by the *dot*, which also represents the geometric extreme point. Hence, the first analogy between elements of geometry and elements of arithmetic is established. The second principle, plurality, can not be consistently represented by less than two dots united. This union of indivisible parts has given us the term, "numeric unit", for any numerical figure which is divisible, in contra-distinction to the singular fraction which is not divisible, and therefore can not properly be considered a unit in the scientific sense. The third principle, totality, in its lowest aspect, is properly defined as a sum, or unit, composed of the least plural number, and the prime singular. This, the least total, equals three indivisible fractions, or one unit and one fraction. Hence, the fundamental basis for any scientific numerical system is expressed by the dots:

.	..	...
one	two	three.

It is quite reasonable that notation began in that manner, and by constantly adding, in regular order, a dot to the previous unit, the idea of endless *multitude* occurred. Later on, the idea of *magnitude* ripened in man's mind and was applied to economize time and labor in noting a count. The several single dots composing each successive sum (a unit) were fused into greater dots commensurate to the increased magnitude, and these greater dots were finally translated into distinct numeric figures representing definite magnitudes, thus:

one	two	three
•	●	⬤
1	2	3

From this primary conception of numerical systems, notation has been developed into the cardinal numbers, from which again has sprung the decimal notation.

The cardinal numbers are expressed by the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 0. The last in the series, called a cipher, in conjunction with the prime fraction (1), is called ten (10) and represents one cardinal total. This cardinal total may be multiplied indefinitely and may be extended to a limitless multitude of numeric values, which aggregately represent all possible proportions of anything computable. The eight cardinal units and the prime numeric fraction, represent a total of 45 indivisible parts. This fact will later on interest the student when he learns and understands the constant harmonious relation of numeric figures and geometric forms.

The decimal notation is a departure from the fundamental principles of *rational* numbers. Decimals are obtained by dividing a given lesser value by a given greater value, after a cipher has been added to the given lesser value, thus:

3 and 2 are given values

```
3 ) 20 ( 666
    18
    --
     20
     18
     --
      20
      18
      --
       2
```

This process of adding a cipher to the remainder and divide by the given divisor, produces a quotient of interminable value, as shown by the figures 666 continued without limitation.

These figures, however, are not whole numbers. They are fractional values of tens and multiples of tens, and properly expressed, they should be formulated as the fraction $\frac{666}{1000}$th.

Hence, all decimals are, like *vulgar* fractions, composed of a *denominator* placed under the line which denotes the magnitude of a given whole of which the *numerator* placed above the line represents a number of parts. For the sake of economy, the denominator is wholly omitted, but is represented by a dot placed before the first figure of the numerator, thus: .666

Decimal notation is an erratic expedient. For conventional use in counting indefinite items and for computation of variable and arbitrarily given values of things, the decimal notation answers very well; but, geometric values, which are inherent in the relation of all the geometric elements, must be computed by constant measures of proportion, called *ratios*. This computation by ratio involves the use of differential unit-measures fitting each element computed, therefore, the decimal system can not be applied to geometry, because, decimals have a constant common unit-measure not fit to represent or express geometric proportions. Numbers fitting the geometric proportions, when tabulated as ratios, are called commensurational numbers, and the code of rules and operations by which the commensurational numbers are obtained, constitute what is called, Commensurational Arithmetic

SECTION B.

Part I.

TERMINOLOGY

1. As the principles of geometry apply to measurement and balance of everything computable within the comprehension of man, easy application and economic methods are essential to satisfactory results. To that end it is necessary to introduce and have accepted fixed definitions of certain conventional terms not hitherto properly defined in current dictionaries, besides giving the technical terms and symbols, usually employed in a course of mathematical study. Hence the following definitions.

FUNDAMENTAL DEFINITIONS.

2. Time — is a measure of duration and change. The time-measure which most concerns the mathematician, is the constant change of the sun's apparent location in the firmament. It is to this phenomenon we trace the first ideas of making clocks and watches, which, are imitations of the great astral dial which measures the time it takes planetary bodies to move from one point in the visible universe to another.

3. Space and **Expanse.** Apparently unoccupied localities, bounded by definable limits, constitute space. For example: the punct made by a material point exhibits space after the point which occupied it is withdrawn. Likewise, the sphere shows the space fitting a globe, while the globe defines the form of the space it occupies. Hence, space may be either vacant or occupied. But in either case, space must be marked by a boundary. The

greatest boundary to space which is subject to man's observation, is measurable only by individuals' range of vision aided by the most powerful telescope obtainable. Beyond this visible bounding limit set by nature, conception is lost in the idea of one ulterior vast illimitable expanse. Hence, it may be said, that space is some part of omnipresent expanse, defined in extent by visible objects which mark some limit. And it follows, from a physical standpoint, that expanse begins were visible space terminates.

4. **Quantity** and **Plenum.** Whatever occupies space constitutes a given quantity which can be measured. Whatever occupies expanse is an unknown quantity called the plenum which is not measurable. Accordingly, symbolic marks, as lines, curves and solids which represent geometric elements and forms, are properly constituted quantities subject to geometric measurements.

5. **Magnitude** and **Size.** Geometric magnitude must not be confounded with the conventional meaning of size. It is shown in exhibit eleven, § 20, Part II, Section A, page 10, that geometric quantities, such as diameters, chords, sines, etc., have no other than inherent relative values which can not be changed arbitrarily. The number of parts into which a geometric quantity shall be divided, according to natural law, is called a geometric magnitude, and this magnitude, in conjunction with the quantity's form, gives identity to the quantity irrespective of size. In computation, geometric magnitudes are numerically expressed, and the mumerical figures which express the magnitudes are called commensurational numbers.

6. **Science** and **Geometry.** Science is the stored up record of universal principles and natural law which from time to time are discovered and applied to the re-production of pheno-

mena. We recognise science when the geometer produces in miniature the horizontal circle ; science is manifested when the electrician produces flashes like lightning in the sky ; and again, we see science when the chemist produces artificial ice. The principles of all science is concentered in pure geometry. The principles are found in every branch of physics. We find them in chemistry, in mechanics, in astronomy, etc. But physics differ from the parent-source in this, that no limit can be set to physics, because, new combinations, new phenomena, and higher and lower grades of beings are among the possibilities of existence, hence, physics can not be exhaustively treated as a whole, while geometry, which is the science of first principles, can be exhaustively treated, since these first and universal principles are numbered and never change, however limitless may be the application of them to a numberless variety of doings and things. So positive and stable is everything pertaining to geometry, that nothing is found impossible in its domain, neither is there room for speculation or negation in geometric discussion.

7. **Geometric Truths.** Geometric Truths rest on proven facts as well as on mental operations and rational conclusions. But, as the relative meanings of "facts" and "truths," conventionally speaking, are not often discussed, a brief explanation which shows the difference, may lead to a more satisfactory understanding. Truths relate to thoughts and are evolved by mental operations and contemplation of mind. Facts relate to doings and things observed and are established by ocular demonstrations. Examples : One person expresses to another, through use of language, what he thinks and knows about a given subject. If the other person recognizes his own thoughts in the first person's expression, a common-sense understanding is proven. In that case a mutual agreement is found which proves that one

mind is balanced by another. Hence it may be said that *truth* is a balanced measurement of mind recognised as truth by all people with a like understanding. A fact is also a balanced measurement, but the measuring-means and that which is measured are different, as for example: a salesman measures tape with a yardstick and obtains by comparison, a length of tape equal to the measure (a yard). This balanced measurement proves to his vision, without any reflective operation of the mind, the fact, that a yard of tape is given ; and the same fact is admitted by all who sees the measurement, in so far, as they agree on the equality of the two lenghts.

The totality of "geometric truths" which underlie mathematics are not established by mere acceptance or recognition of a few persons, while others differ and *believe* something else, unless those who differ, fail to give facts which prove the contrary. Geometric truths and co-ordinate facts are forced upon man by rulings in nature, which all rational beings must abide by, in order to explain to themselves and to others the use of the science. Everything pertaining to geometry which is proven or can be proven true, by ocular demonstration, by construction and by commensurational arithmetic, is part of the science ; that which is not so proven, is not a part of the science.

Proof of anything in geometry involves the following requirements : first, a proposition whicn affirms that this or that is true. Associated with the proposition must be a geometric diagram or some device representing the geometric aspect of the proposition. Rules for operating numeric figures must be given and followed, so as to properly correlate the arithmetical computation to the principles of geometry. All this must be done and proven harmonious before a geometric problem is satisfactorily solved.

COMMON TECHNICAL TERMS.

8. Proposition and Theorem. A geometric proposition must be a brief statement of apparent facts relative to geometric quantities and principles. The diagram which conveys the meaning of the proposition is called a theorem when the several parts of the diagram and their mutual relations are explained in accordance with the language used in the proposition.

9. Axioms and **Postulates.** A proposition which is self-evident to reason and at the same time demonstrated by obvious facts, is called an axiom. For example: "A part of any given whole is less than the given whole" (This statement is an axiom, since anything cut up into parts illustrates the fact.). Some propositions, however, are self-evident to reason, yet are not susceptible of ocular demonstration. For example: "A line may be extended to any length, but the longest line is nevertheless shorter than the endless uniform curve compassing the longest line" (see scholium in Section A, Part 1, § 44, page 12). The statement in this case is convincingly true to reason, yet, not demonstrable by ocular illustration, since we can not produce and measure the longest line. This and similar propositions are called postulates.

10. Problems and **Solutions.** A geometric problem is a proposition demanding the methods by which certain things can be accomplished and proven possible. The manner and means by which these things are accomplished and proven possible, constitute a solution of the problem.

11. Corollary and **Scholium.** Corollary is the name for a statement of facts deduced from a foregone proposition already proven true. Scholium means explanatory remarks relating to something said or done, which prevents misunderstanding.

12. Lemma—means: something demonstrated and proven true which should be known in order to make easier the demonstration of something else to be proven.

IDIOMATIC TERMS.

13. In geometry the following distinction is made as to the meaning of the words ***on*** and *over*, *around* and *under*, when these words are used in connection with geometric figures and forms. A square ***on*** the line means a square plane, the side of which is the line. A square *over* a given line means that the given line is either the diagonal of a square, or, the given line bisects the square's plane into two equal rectangles.

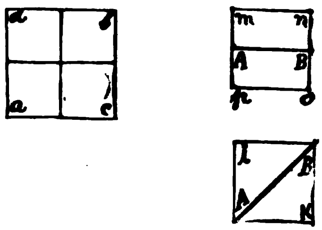
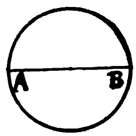


EXHIBIT I.

The square *a b c d* is on the line *a c*.

The squares *m o n p* and *A B i k* are squares over the line *A B*.

EXHIBIT 2.



Circles are always described over a line—the diameter; hence, the circle over *A B*.

In a diagram the bounding lines and curves which mark the boundaries of space, constitute "figures" in opposition to the spaces under the figured outlines, which constitute "forms". In volumes or bulks, as the sphere and the globe, the hollow of the sphere is a form, while the solid globe in and under the sphere's boundary is a figure.

14. A geometric line is not squared in the same manner as a geometric root,* which, multiplied by itself gives the square. Squaring a given line, means in geometric terminology, that the line is divided into four equal parts, each of which form the side of a square perimeter. Likewise, a circle's circumference can be cast into the form of a square perimeter after each quadrant-arc has been rectified (or straightened out). But neither the squaring of a circle's circumference or the squaring of a line, as described, must be confounded with squaring of the numeric magnitudes of these elements.

If, for example, the magnitude of a radius is given as 9, the square root of that radius is 3, and is represented by one-third of the radius. If the magnitude of a given radius is 3, the magnitude of the radius may be squared by subdividing each of the three parts composing the radius, into three minor parts.

It will be seen that the conventional sense of "squaring" does not apply to linear values representing linear magnitudes of extent. In that connection, "squaring" means multiplication of the given value representing the magnitude, so that, a number of minor subdivisions are obtained equivalent in their total to the major divisions of which they are parts.

If, for example, a line of the magnitude 6 (units) is given, then, if each of these six units are subdivided into 6 nits, then it may be said, that the given line has been squared, whereby, 36 linear nits of equivalent value to the first given 6 units are produced.

So on, these 36 nits may again be squared and converted into 1296 equivalent finits, by simply subdividing each of the given nits into 36 finits.

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* See § 58, Page 24, Section A, Part III.

ARITHMETICAL TERMS.

15. The principal operations and corresponding signs used in arithmetic, are: addition (+), substraction (—), multiplication (×), division (÷), and equational proportion governed by the so-called "rule of three."

The sign of addition is called *plus;* the sign of substraction is called *minus;* the result obtained by the process of adding figures together is called a *sum;* the result obtained by the process of subtracting something less from something greater, is called the *difference.*

Examples: o+oo gives the sum ooo, or 1 plus 2 makes 3. ooo—oo gives the difference o, or 3 minus 2 leaves 1.

The sign of multiplication is called *times*, and the result obtained by the operation is called the *product.* Multiplicatlon is an economic manner of adding quantities together, which the following examples show:

ooo+ooo+ooo gives the sum ooooooooo. (9)

ooo × ooo gives the product ooooooooo. (9)

The sign of division signifies that one given quantity shall be *measured* by another given quantity. The result obtained by this operation is called the *quotient.* The quantity given to be measured is called the *dividend;* the given measure is called the *divisor*, and the quotient obtained, tells how many times the divisor is contained in the dividend

Thus: ooooooooo÷ooo gives the quotient* ooo.

The sign of equalization (=) means "equal to" and is substituted in computation for the words: gives "the sum", "the difference," "the product," "the quotient"; thus:

$$1+2=3;\ 3-2=1;\ 3\times 3=9;\ 9\div 3=3.$$

*See Scholium, § 15, Part II, Page 7, Section A.

MATHEMATICAL FORMULAS.

16. Any arithmetical process or operation performed and expressed by numeric figures, symbols and signs, instead of by conventional language is called a formula; hence it may be said, that "formula" means a prescribed mode of operation expressed in the simplest manner.

EQUATIONS.

17. A balanced measurement of values obtained by a given formula, is called an equation. The simplest kind of equations are expressed by two terms connected by the sign of equalization, thus: ○=○. But, mathematical equations are not balanced by the principle of equality alone, the principle of equity* or proportional difference must be involved in the balance. For example, $2=2$ is not an equation in the scientific sense, since both terms are the same and express nothing but tautology. The terms $1+2=3$, express an equation, since a proportional difference of one of the terms is involved in the balance, that is, the term $1+2$.

Where two or more factors compose a term, the term is called composite, to distinguish it by name from single terms. For example: the term 3 is a single term, while $1+2$ is a composite term. In simple equation, comprising only two terms, both the terms may be composite, as: $2+4=2\times 3$.

In such cases, it is proper to place a line above the several factors composing a term, or, inclose the whole term in brackets. For example:

$$\overline{2+3}=\overline{4+1}\,;\ (\overline{2\times 3+1}=7)=(\overline{8-1}=7).$$

$2\ (4)=\overline{(3\times 3)-1}$, means: twice four, equals three threes less one.

*See Commensurational Division, §§ 12—20, Section A, Part II, Pages 5—10.

EQUATIONAL PROPORTION.

18. The formula of compound equation or equational proportion is governed by the "rule of three". The rule of three is so-called, because, in the operation, three terms are given by which a fourth is obtained. For example: 1 2 3 are given terms. Required, a fourth term, which shall be related to the third as the second term is to the first. But the relations of the first and second term may be one of difference, or the relation may be one of proportion. If it is one of difference, a horizontal colon (··) expresses that relation. If the relation is one of proportion, a vertical colon (:) expresses that relation.

Hence: o ·· oo :: ooo ·· oooo (difference of 1 and 2 equals the difference of 3 and 4), or: o : oo :: oo : oooo (proportion of 1 and 2 equals the proportion of 2 and 4).

19. The rule for finding the fourth term when the first three are given, when the relation is one of difference, is: add the 2nd and the 3rd terms and subtract from the sum the 1st term, the difference equals the fourth term sought. Thus:

$$
\begin{array}{l}
1 \cdot\cdot\, 2 :: 3 \cdot\cdot\, 4 \\
\quad\;\, + \\
\quad\;\, \underline{3} \\
\quad\;\, 5 \\
\quad\;\, \underline{1} \\
\quad\;\, 4 = \text{the term sought.}
\end{array}
$$

When a diffe tial equation is given, as above, the correctness of the figures is te ed by adding the 1st and the 4th terms together and by adding the 2nd and the 3rd together; if the two sums are equal, the equational balance is proven, thus:

$$\overline{1+4}=5 \text{ and } \overline{2+3}=5.$$

20. The rule for finding the fourth term when the first three are given, when the relation is one of proportion, is: multiply the 2nd and the 3rd terms and divide the product by the 1st term; the quotient equals the 4th term sought. Thus:

$$1 : 2 :: \overline{2 : 4}.$$
$$\overline{2 \times 2} = 4 \qquad \overline{4 \div 1} = 4.$$

When a proportional equation is given, as above, the correctness of the figures is tested by multiplying the 1st by the 4th term and the 2nd by the 3d term; if tne two products are equal, the equational balance is proven.

NUMERIC FRACTIONS.

21. Numerical fractions are of three kinds, viz.: Proper fractions*, improper fractions and decimal fractions. Decimal fractions have always the same denominator (ten or multiples of ten). As this is understood, the denominator of the fraction is omitted, and in its stead, a dot is placed before the first figure of the numerator, thus:

$^5/_{10}$ is written .5, which denotes that the given value is one-half ($^1/_2$).

22. Proper fractions have variable denominators which always are greater than the numerators. For example: $^2/_3$ or $^3/_7$ are proper fractions, which denote parts of a unit.

23. Improper fractions have greater numerators than denominators. For example: $^5/_2$ or $^7/_3$ are improper fractions.

24. In all cases, the denominator denotes the magnitude of a given unit, hence it follows, every improper fraction represents one or more units and a fraction of a unit. For example:

$$^7/_3 = \overline{2\,(3) + {}^1/_3} \text{ of a unit.}$$

Or it may be expressed: $2^1/_3 = \overline{2 \times 3 + 1}$ or $^7/_3$.

*Also called vulgar fractions.

SECTION B.

GEOMETRIC RATIOS

AND

COMMENSURATIONAL FRACTIONS.

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25. Ratios are measures of proportion expressible in whole numbers.

It requires two numeric values to express one ratio.

Each of the two numeric values must be composed of parts which are aliquot divisors for either of the numeric values.

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad E \\ A\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot B. \quad C\cdot\ \cdot\ \cdot\ \cdot D. \end{array}$$

For example : A. . . . . B. C. . . . D.

A B and *C D* are given lines the relative magnitudes of which are required, so that the ratio of the two lines can be given. To obtain the ratio, the lesser line must be used as a dividing unit-measure for the greater line. Demonstrate with compasses.

$A\ \overline{B \div C}\ D = 2\ \overline{(C\ D) + \tfrac{1}{3} \text{ of }}\ C\ D$. That is to say: the lesser line *C D* is contained twice in the greater line *A B* and leaves the fraction *E B* which is equal to one-third of *C D*. In this case it is evident that the fraction *E B* is a common measure for both the given lines; hence, "commensurational fraction" is the given name for all similar common measures which are parts of the geometric unit. The measure of proportion, that is, the ***ratio***, is expressed by the two magnitudes $3:7$. These figures, however, do not express the unit-values of *C D* and *A B* when *C D* is the unit-measure. The unit-values are $C\ D = 1$ and $A\ B = 2\tfrac{1}{3}$.

NITS.

26. The foregoing explanation of commensurational fractions' relation to geometric ratios, makes it plain to the understanding that geometric lines can have two distinct values commensurationally adjusted, each to the other. One, is the ***rational*** value made up of commensurate fractions, called ***nits***; the other, is the unit-value, each unit of which, is composed of a number of nits which give magnitude to the unit-measures. For example: A given

unit's magnitude is 3 nits (represented by the line $C\,D$). The unit-value of $A\,B$ is then $2\frac{1}{3}$ unit. The commensurational adjustment of these divers values (units: $2\frac{1}{3}$, nits: 7) is explained by converting the unit value into multiples and the nit value into units, inasmuch as, $2\frac{1}{3}$ equals 7 multiples, and $7 \div 3$ equals $2\frac{1}{3}$ units.

By applying the rule of three, the equational proportion of the two values are proven, thus :

Rational value 3 and 7.
Unit value 1 and $2\frac{1}{3}$.
By ratio : $3 : 7 :: 1 : 2\frac{1}{3}$.
Since ; $(7 \times 1) = 7$ and $(3 \times 2\frac{1}{3}) = 7$.

27. Finally it follows, that the commensurational fraction determines what the magnitude of the proper unit-measure shall be in any geometric computation, since the denominator of the fraction always equals the unit's magnitude.

For example : A——————$B\overset{5}{\underset{7}{——————}}C.$

If $A\,B$ is a given unit and $B\,C$ is $^5/_7$ part of $A\,B$, the magnitude of the unit $A\,B$ shall be 7 nits, and the fraction $B\,C$ shall be 5 nits ; while the ratio of $A\,B$ and $B\,C$ is, $7 : 12$.

By equation it is shown, that

$$7 : 12 :: 1 : 1^5/_7.$$
Since $7 \times 1^5/_7 = 12$ and $12 \times 1 = 12$.

28. This same principle of differential values, relating to lines also applies to planes, as the following exhibit shows.

EXHIBIT 3.

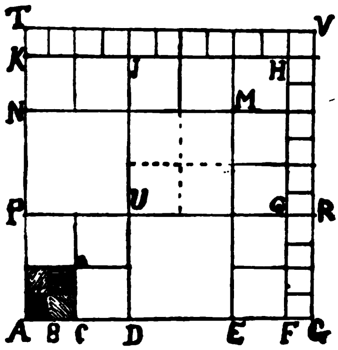


Exhibit 3 shows an integral square ($A\ M\ N\ E$) of the magnitude 4 units, one of which is marked $A\ U\ P\ D$. In this case the given unit is an aliquot divisor for the given square which it divides into four equal parts. This same unit ($A\ U\ P\ D$) is not an aliquot divisor for the greater square $A\ H\ K\ F$; and in order to find the proper unit-measure for this greater square, the proportion of $E\ F$ to $A\ E$ must be given. Now, in this case, it is found that $E\ F : E\ A :: 1 : 4$. Hence, when the linear unit $A\ D$ is bisected at C, two linear nits are given as $A\ C$ and $C\ D$ either of which equal the fraction $E\ F$. By constructing square planes on these two nits, a magnitude of 4 nits is given to the unit $A\ U P D$. And now it is found that one of these areal nits, as $A\ O$, is an aliquot divisor for the greater square $A\ H\ K\ F$. It is shown in the exhibit, that the difference of this greater square and the lesser square first given, equals 9 areal nits; and as 4 nits $=$ 1 unit, the difference in the magnitudes of the square $A\ M\ N\ E$ and the square $A\ H\ K\ F$, computed by the given unit $A\ U\ P\ D$, is $2\frac{1}{4}$ unit, or 2 units and 1 nit.

Hence, the relative magnitudes of the two given squares (*A M* and *A H*) are: 4 and $6\frac{1}{4}$. To express the ratio of the two squares, both must be reduced to nit values, in which case, the lesser square is represented by the number 25. The commensurational adjustment is proven by the equation :

$$16:25::4:6\tfrac{1}{4}$$

$$16\times 6\tfrac{1}{4}=100 \qquad\qquad 4\times 25=100$$

This substitution of nit values for unit values, in order to obtain rational relations expressed in whole numbers, calls for a sign expressive of equivalence, viz: *e. v.* Thus: *e.* nit *v.* means, equivalent nit value, or a number of nits equal in area to a given number of areal units.

FINITS.

29. If the magnitude of the last given square is augmented by adding the linear fraction *F G* to the side, the square *A V T G* is given; and now it is found that the previously given nit-measure will not divide that greater square evenly. But, if the given nit is properly subdivided, an aliquot divisor can be found. Subdivisions of nits, which are aliquot divisors for the greater square's plane, are called *finits.*

For example: In the exhibit 3 the linear finit is represented by *A B* and by *F G.* The corresponding areal finit (which is a square plane constructed on the linear finit) is shown by the plane *A I.* Now it is found, that 21 of these finits equal the fractional plane *F G V T K H F*, and as in this case, four of these finits are required to make one nit-value, and as four nits make one unit-value, it follows, that 16 finits equal 4 nits or 1 unit. Accordingly the *elbow*-formed plane *F G V T K H F*, equals $1\,{}^{5}/_{16}$ unit ; and the whole square *A V T G*, contains :

$$(4 \text{ units}+2\,{}^{1}/_{4}\text{ units}+1\,{}^{5}/_{16}\text{ unit})=7\,{}^{9}/_{16}\text{ unit.}$$

Converting this total unit value into multiples, by multiplying the unit by the denominator of the fraction and by adding to the product the fraction's numerator, we obtain:

$$\overline{7\times 16+9=121}\text{ finits.}$$

Counting the linear finits contained in the side *A G* of the greatest square *A V T G*, we find eleven of these fractions. This shows the commensurational relation of area and perimeter, obtained by commensurational division, since $11 \times 11 = 121$ gives the *e.* finit *v.* of the greatest square in the exhibit 3.

The adjustment of the given squares *A M N E*, *A H K F*, *A V T G*, through the use of fitting unit-measures, illustrates the general principles which underlie the science of commensurational arithmetic and commensurate ratios.

30. In the process of the above given commensurate adjustment three equivalent values occur, namely :

Lateral unit-value $2\,{}^{3}/_{4}$; areal unit-value $7\,{}^{9}/_{16}$.

Lateral nit-value $5\,{}^{1}/_{2}$; areal nit-value $30\,{}^{1}/_{4}$.

Lateral finit-value 11 ; areal finit-value 121.

To still further economize space and labor in expressing formulas, the unit is abbreviated to *U.*, the nit is abbreviated to *nt.*, the finit is abbreviated to *fnt.* "Equivalent nit-value" is thus expressed : *e. nt. v.* and "equivalent finit value" is expressed : *e. fnt. v.* The sign (+) placed after *U.*, signifies a ***major*** unit in opposition to the sign (−) placed after *U.*, which signifies a ***minor*** unit. Thus: *U* + and *U* −.

31. In order to distinguish between the lateral value (the magnitude of a square's side) and the square's root which is part of the plane bounded by the perimeter, two signs are used, viz : $\sqrt{}$ and $.\sqrt{}$.

$\sqrt{}$: Signifies root of an integral square.

$.\sqrt{}$: Signifies side of a surd square or the lateral value.

In order to distinguish between the lateral value of a cube (the magnitude of a cube's edge) and the cube's root which is part of the cubes bulk or volume, two signs are used, viz : $:\sqrt{}$ and $3\sqrt{}$.

$:\sqrt{}$: Signifies edge of cube.

$\cdot 3\sqrt{}$: signifies root of cube.

SEVERAL OTHER SIGNS USED IN COMPUTATION.

32. A small figure ^{2} placed after a given numeric value, signifies that the given value shall be squared, or: a square shall be constructed on a line of given magnitude and the corresponding magnitude of the square's plane shall be given.

Thus: $2^2=4.$ $(2\frac{1}{2})^2=6\frac{1}{4}.$

A small figure ^{3} placed after a given numeric value, signifies that the given value shall be cubed, or: a cube shall be constructed on a line of given magnitude and the corresponding magnitude of the cube's volume shall be given.

Thus: $2^3=8.$ $(2\frac{1}{2})^3=15\frac{5}{8}.$

MEAN-PROPORTIONALS.

33. A mean-proportional value is given when the product of the 2nd and 3rd terms of any equational proportion equal the product of the 1st and the 4th terms.

Thus: $2:4::4:8.$

That is: $2^2=4.$ $4^2=16.$ $8^2=64.$

Hence, 4 is the mean-proportional of 2 and 8, and, 16 is the mean-proportional of 4 and 64.

The abbreviated expression for "mean-proportional" is *mp*

The abbreviated expression for "magnitude" is *mg.*

The abbreviated expression for "difference" is *dff.*

The abbreviated expression for "number" is *No.*

The abbreviated expression for "radius" is R.

The abbreviated expression for "generating factor" is *Gf.*

The symbol for circumference of circle is ○.

The symbol for circle's plane is ⊙.

The symbol for perimeter of square is □.

The symbol for plane of square is ■.

The symbol for inscribed trihedron is △.

This symbol: $\frac{\bigcirc}{4}$, means quadrant arc.

This symbol: $\frac{\odot}{2}$, means semicircle.

This symbol: $\genfrac{}{}{0pt}{}{\bigcirc}{2}$, means semicircumference.

The Greek letter π stands for ratio of diameter and circumference of circles, and is called "*pi.*"

Dmt. stands for diameter of circle.

Dgl. stands for diagonal.

The signs: $>$ and $<$, stand for "greater" and "less." When placed between two geometric elements, the element nearest to the point is less than the element nearest to the opening, thus: the arc of a segment is greater than the chord of the segment ($A > C$).

SUMMARY OF PART I., SECTION B.

NUMBERS:

Notation — Systems — Fractions — Units—Totals—Proper Fractions—Inproper Fractions—Decimal Fractions—Commensurate Fractions—Nits—Finits.

GEOMETRIC RATIOS:

Equations—Equational Proportion—Mathematical Formulas—"Rule of Three"—Mean-Proportionals.

RATIONAL TERMINOLOGY:

10 Fundamental Terms—9 Technical Terms—4 Idiomatic Terms—25 Arithmetical Terms—19 Abbreviated and Symbolic Terms—Supplement.

SUPPLEMENT.

SUGGESTIONS TO THE TUTOR.

Section B of Rational Mathematics differs from section A, in this respect, that the subject-matter of the former calls into activity mental operations of the studious order rather than the functions of the perceptive faculties through which we observe what appears and is done outside the mind, as shown by the manner of treating the subject-matter of the latter. But, it is not enough for the student to memmorize and recite the several definitions and terms which comprise the terminology of rational mathematics ; the full meaning and the proper application must be understood. To convey that knowledge, in an acceptable manner, it is necessary to adopt systematic methods of instruction. By suitable object-lessons, the intelligent tutor can make this initial study of mathematical terminology interesting as well as instructive. For example : In the definition given of "time," the "astral-dial" might puzzle a dull scholar, but, if the tutor will represent the stars in the firmament by a few chalk marks on the black-board and will describe the apparent course of the sun as it moves from one star to another, the comparison of these chalk marks, to the figures on a watch dial, and the course of the sun to the course of the clock's pointer, as it passes from figure to figure, which mark the hour, is readily recognized even by a dull scholar. To explain the difference in the meaning of the terms "space" and "expanse," and "quantity" and "plenum," which to some people appear too abstruse for juvenile comprehension, let a circle be drawn on a black-board, and point out the circle's circumference as a representative mark for the limit to an individual's perception when aided by the most

powerful telescope, and from this bounding limit draw radiating lines representative of illimitable extension beyond one's power of observation, then, even the dull scholar can be made to understand that an omnipresent field exists of which we only can observe so much as our organs of vision will allow. This explains, that any part of this omnipresent field to which we can discern a boundary, is called space, in contra-distinction to that part of the omnipresent field which is beyond the visible and is called expanse. That much understood, it is easy to explain the difference in the meaning of quantity and plenum, since whatever is contained within space is called quantity in contra-distinction to whatever is contained in the domain of expanse and is called plenum. Plenum, of course, must forever remain occult to the human comprehension so long as our observing faculties are limited by incapacity; and it follows, that whatsoever is contained in that occult domain is not measurable or computable and need not be considered in connection with mathematical study otherwise than having it pointed out to exhaust inquiry about the source of spacial capacity and geometric quantity.

By similar illustrations and painstaking on the part of the tutor, these apparently abstruse subjects are reduced to simple propositions, the study of which, develop spontaneously the faculties of comparison and reason and lays a mental fundation to sound philosophical inquiry.

The origin of numbers, as presented in the preface to Part I, Section B, can be made an interesting subject for discourse, which will dispel much of the prevailing ignorance about numbers generally and will expose some of the irrational theories on which modern notation is founded. It should be made clear to the pupil's understanding that numbers, symbolically expressed by figures, are simply invented devices designed by human agency to represent

quantities and magnitudes, as the letters of the alphabet are devised symbols designed to represent sound and sense.

Particular efforts should be made to show why a decimal notation is not fit for geometric computation; and it may be shown to advantage, that letters of the alphabet as well as the numeric figures are all derived from and composed of the three first elementary symbols in geometry, viz: dot, line, curve. The object of drawing attention to this fact, is to show the natural priority of geometry.

It will be noticed that the terminology, in Part I, has been divided up into groups of terms and symbols, and each group has been designated in accordance with the special part of mathematics to which the group applies. The terms or symbols in each group have also been numbered for the purpose of aiding the memory and for easy reference when one is called upon to describe a symbol or define a term.

It is excellent exercise to have a pupil express on the blackboard by symbols and abbreviations a statement gived in conventional language; and likewise, formulas may be given by the tutor which the pupil shall translate into conventional language. These and similar exercises soon show who have studied and understood and who have not.

SECTION B

PART II

PREFACE.

(TO THE TUTOR.)

When from discoveries or general intellectual progression, certain educational changes occur, which make it necessary to alter the old methods of teaching and to introduce others better adapted to the requirements of the new education, it is proper to discuss, in advance, the difference as well as the analogue. Hence, these remarks are directed to what may be called **analytic analogy** between the old and the new mathematics. Mathematics comprise Geometry and Arithmetic in unity; but, the true relationship between geometric forms and arithmetical figures has not until recently been established; hence, mathematics has been considered an imperfect science (?) One of the causes for this seeming imperfection is found in the irrational and interminable notation (decimal notation) which has been in use for many centuries. The result is, that two kinds of mathematics are in the field. One is commensurable in principle, while the other is incommensurable. The former is based wholly upon measures of proportion (ratios) as already have been defined and explained in Part I, Section B, and for this reason that kind of mathematics are called: rational mathematics, in opposition to the latter kind, which should properly be called irrational mathematics, were it not, that the name "calculus," has for a long time been used as the proper term for irrational calculation. Calculus is based on false premises and is carried on by sophistical reasoning. One of its fundamental premises states, that the difference between an arc and its subtending chord may be so trifling, that these two elements are considered equal in regard to extent. Not being able, from these and similar premises or by equally irrational methods of procedure, to obtain the exact relative values required to prove

anything absolutely true, modern mathematicians of the Descarte's school trust entirely to approximate results obtained by algebraic processes, which may be extended infinitely to neverending series of transcendental calculations.

The kind of algebra used in calculus is derived from the ancient Persians and Arabians. Among these Eastern people the art was used by the Magii or High-priests as a mystic symbolism to inspire awe for their power, based on learning which no one could understand other than those initiated in secret lore. But, when stripped of its fascination, algebra is simply a short-hand method for expressing mathematical formulas, and unless these formulas are translateable into exact numeric values, they are of no more use to science than stenographic notes, which can not be translated into readable language, are to literary economics.

No other than the simplest form of algebra which is translateable into exact and finite notation is employed in rational mathematics, and for the purpose of avoiding misunderstanding, this kind of algebra is named: technical algebra.

WHY THESE CHANGES ARE NECESSARY.

The efficient causes for the necessary changes in the method of teaching mathematics, as alluded to in the foregoing remarks, are briefly embodied in the twenty-eight articles following and which comprise the recent findings and discoveries on which the new geometry and commensurational arithmetic is founded.

It is found : **1st,** that the right sine of a sextant equals one side of the inscribed heptagon; **2d,** that the chord subtending $^2/_7$ of a circle's circumference equals in extent the quadrant arc. This chord is named the rectifying chord. **3d,** that the equal square to circle is a mean proportional between square on diameter and a square equal in perimeter to the circle's circumference. Hence, the rectifying chord is a factor in squaring the circle. **4th,** that

the side of equal square to circle, equals a chord subtending $^{6}/_{17}$ths of a circle's circumference. This chord is named the squaring chord, and is a factor in squaring the circle. **5th,** that a chord subtending $^{73}/_{240}$ths of a circle's circumference equals in extent the edge of a cube, the volume of which cube equals the volume of a sphere constructed over the circle in which the chord is drawn. This chord is named the cube chord, and is a factor in cubing the sphere. **6th,** that the square on radius, the square on quadrant chord, the square on tridrant chord, and the square on diameter, are respectively proportioned, as are the numbers 1, 2, 3, 4; **7th,** a right-angled triangle, the squares on the perimeter of which are proportioned as 1, 2, 3. **8th,** that the perimeter of the triangle (No. 7) can be formed by radius, quadrant chord and tridrant chord in any circle; **9th,** that the perimeter of the triangle (No. 7) is proportioned in its several parts as are the edge of a given cube. the diagonal of face of the given cube, and the diagonal of the given cube's volume. **10th,** a right-angled triangle, the squares on the perimeter of which are proportioned as are the numbers 2, 3, 5. **11th,** a right-angled triangle, the squares on the perimeter of which are proportioned as are the numbers 1, 4, 5. **12th,** a right-angled triangle, the squares on the perimeter of which are proportioned as are the numbers 1, 8, 9; **13th,** that the triangle (No. 12) added to the triangle (No. 7) forms the vertical section of a given tetrahedron, the altitude of which equals the cube chord in a given circle, while the major slant of the given tetrahedron equals the diameter and the minor slant equals the tridrant chord in the given circle; **14th,** that the altitude of a tetrahedron constructed on the inscribed trihedron in a given circle equals the quadrant chord in the given circle; **15th,** simple methods for trisecting lines, arcs and angles; **16th,** that the right sine of a hexagon, or the side of a heptagon, is a factor in doubling the

cube; **17th,** a simple method by which the lateral proportions of squares, evolved in the order of the cardinal numbers 1, 2, 3, 4, 5, 6 and so on, can be definitely determined; **18th,** that faces of doubled cubes are proportioned in inverse ratio to squares evolved in the cardinal order as stated (No. 17).

Discovered: **19th,** that geometric elements and forms have distinct and fixed measures which cannot be arbitrarily changed, and that these measures change their values according to the elements' relation to different forms and the difference of the forms compared; **20th,** the difference of increment and decrement relative to areas contained under curves and lines of equal extent; **21st,** a principle in fluxions whereby surds are converted into integrals by natural unit measures, named solvents; **22d,** a universal law which governs the finding of the natural solvents for surd squares; **23d,** a universal law which governs the adjustment of lateral values and areal values of squares, aside from "roots"; **24th,** a commensurational cube within a sphere, which in its functions is an aliquot divisor for all the sphere's dimensions. This cube is named the "pi" cube; **25th,** a true pi value correlated to and evolved from the pi cube; **26th,** a true and constant pi-formula; **27th,** a trigonometrical canon evolved from the finding (No. 17); **28th,** that all geometric elements are commensurately related to each other, and all to the true pi value.

SECTION B.

Part II.

TECHNICAL ALGEBRA.

1. Technical algebra expresses in the most economic manner, by signs, symbols and single letters the arithmetical operations which ought to be performed on certain given quantities in order to obtain results which determine the relative values of the given quantities to other quantities sought.

Italics are generally used.

2. Single letters may be substituted for two or more of the letters employed to define a geometric form or diagram, thus:

The square $A\ B\ C\ D$ may be expressed by the single letter a, provided it is stated in advance that $A\ B\ C\ D = a$. This substitution of one letter for several, constitutes a common principle of economy in algebra.

As a general rule, when the exact magnitude of a geometric quantity is given, the quantity is expressed by the first successive letters of the alphabet; but when the quantity treated of is indefinitely extended, then, the last letters of the alphabet are used, thus:

$A\ x$ represents a line, the beginning of which is given, but its exact extent is not defined by magnitude. In such cases the whole line may, by substitution, be expressed by either x, z or y.

3. When many different elements occur in a geometric composition, which necessitates repetition of the same letter, an inverted comma suffixed to one of the letters makes the distinction, and this distinguishing mark is called "prime."

EXERCISES IN APPLIED ALGEBRA.

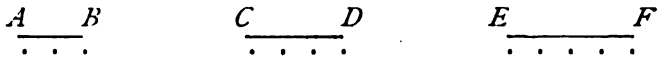


EXHIBIT I.

4. *A B*, *C D*, *E F* are three given lines of the respective magnitudes 2, 3, 4.

By substitution of *a* for *A B*, of *b* for *C D*, of *c* for *E F*, these three elements and their magnitudes may be represented by *a*, *b*, *c*.

Now, if the geometric relations of these quantities shall be proven true by arithmetical operations, the following exhibit explains the method by what is called: formulas.

$$*(A\,B :: C\,D :: E\,F) = (a \,.\,.\, b :: b \,.\,.\, c)$$

Given geometric quantities. Single letters as substitutes.

Translated into numeric values, we obtain:

Algebraic: $a \,.\,.\, b :: b \,.\,.\, c$

Arithmetical: $2 \,.\,.\, 3 :: 3 \,.\,.\, 4$

In conventional language, these letters and figures with the intervening signs, mean:

1st. The line *A B* is related to the line *C D* as the line *C D* is related to the line *E F*.

2d. The difference of *A B* and *C D* equals the difference of *C D* and *E F*.

3d. The sum of *A B* and *E F* equals twice *C D*.

---

5. If on the lines *A B*, *C D*, *E F*, three integral roots are constructed and three integral squares are raised on these roots, we obtain the three square magnitudes 4, 9, 16.

These squares and their magnitudes are algebraically and arithmetically expressed as:

$$a^2 = 4; \quad b^2 = 9; \quad c^2 = 16.$$

* When only three terms are noted, use double colons.

But it must be observed, that the relation of

$$a \,..\, b :: b \,..\, c$$

is not the relation of

$$a^2 \,..\, b^2 :: b^2 \,..\, c^2$$

Since: $4 \,..\, 9 :: 9 \,..\, 16$ is not an equational proportion as is $2 \,..\, 3 :: 3 \,..\, 4$.

This exposition shows, that lines of given magnitudes which form sides of squares constructed on the given lines, are not proportioned each to the other as are the planes or areas of the squares raised on the lines. Hence, the following axiom is known as the **differential** axiom :

6. *Commensurational relations of square perimeters are inherently independent of the relations existing between the magnitudes of the spaces contained under the perimeters.*

7. If on lines proportioned as 2, 4, 8, square roots are raised, and square areas are constructed, it is found, that harmonious relation exists between the perimeters, the roots, and the areas of the squares, thus:

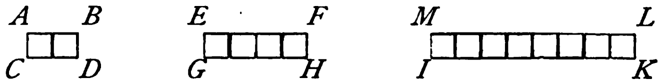


EXHIBIT 2.

A B, *E F*, *M L* are three given lines which may be used as sides for squares, or, which is the same thing, they may be representing one-fourth of a given square perimeter. *A D B C*, *E H G F*, *M K I L* represent three given square **roots** raised on the three given lines.

Now, if we make by substitution: $\overline{AB} = 2 = a$

$$\overline{EF} = 4 = b$$

$$\overline{ML} = 8 = c$$

and, if we make by substitution : $\overline{A\,D\,B\,C} = 2 = a'$

$$\overline{E\,H\,G\,F} = 4 = b'$$

$$\overline{M\,K\,I\,L} = 8 = c'$$

Then: $a'^2 = 4$, $b'^2 = 16$, $c'^2 = 64$. (areal values.)
And: $\sqrt{a'^2} = 2$, $\sqrt{b'^2} = 4$, $\sqrt{c'^2} = 8$. (root values.)
And: $\cdot\sqrt{a^2} = 2$, $\cdot\sqrt{b^2} = 4$, $\cdot\sqrt{c^2} = 8$. (lateral values.)

Reduced to numeric expression, we have:

$$2^2 : 4^2 :: 4^2 : 8^2$$

Since: $4 : 16 :: 16 : 64$.

In this case, equational harmony is shown throughout, and a mean-proportional value is proven by the rule of three.

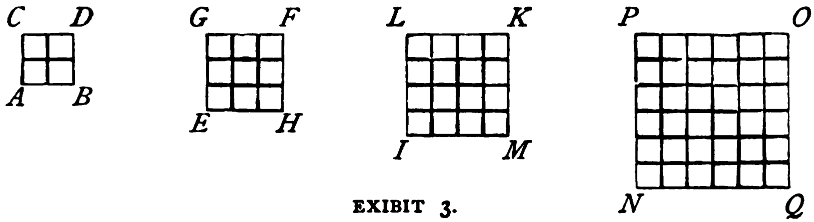
The mean-proportionals, as here proven, are the numbers 4 and 16.

MEAN-PROPORTIONALS.

mp.

8. Mean-proportionals are obtained by geometric construction as well as by arithmetical computation. They are essential factors in production of ratios and serve as adjusting mediums for correlating surd and integral values. Numerically, a mean-proportional is proven when the 2d and the 3d term in a formulated proportion balances the 1st and the 4th terms. Hence, where only three terms are given, as in the exhibit 2, when the middle term is multiplied by itself and gives a product equal to the product of the first and the last term, then, the middle term is virtually squared and represents a mean-proportional square between the squares raised on the values of the first and the last terms.

9. Another mean-proportional relation is presented by the following exhibit:



EXIBIT 3.

By substitution, let a, b, c, d represent the several given square roots in the series, exhibit 3.

Then, $a^2 : b^2 :: c^2 : d^2$

Numeric values: $4 : 9 :: 16 : 36$.

And: $\sqrt{a^2} : \sqrt{b^2} :: \sqrt{c^2} : \sqrt{d^2}$

Numeric values: $2 : 3 :: 4 : 6$.

In conventional language, the meaning of the above given formulas may be stated, thus: A series of squares are given and so proportioned, that the product of the first and the last in the series balance the product of the intermediate squares. But, besides, it is found, that the product of the intermediate squares' roots is the magnitude of a mean-proportional square between the first and the last given squares in the series.

For it is shown, since $a^2 = 4$ and $d^2 = 36$,

That: $(a^2 \times d^2) = (b^2 \times c^2)$

That is: $\overline{(4 \times 36} = 144) = \overline{(9 \times 16} = 144)$

and it is also shown, that:

$\overline{(b \times c} = 12)$ is the mean-proportional square of a^2 and d^2, since:

$$a^2 : \overline{b \times c} :: \overline{b \times c} : d^2$$

or: $4 : 12 :: 12 : 36$.

MEAN-PROPORTIONAL DISTANCES.

10. A geometric mean-proportional distance must not be confounded with the conventional idea of a middle point between two linear extremes.

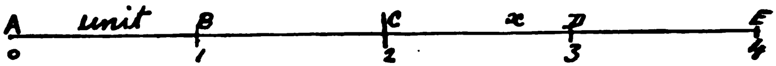


EXHIBIT 4.

For example:

A B is a given linear unit.

A C is a given line of the magnitude 2.

A E is another given line of the magnitude 4; while

A D is a third given line of the magnitude 3.

Apparently, *A D* is a mean-distance between

A C and *A E*. But, applying the rule of three we find the error, since:

$$2 : 3 :: 3 : 4\tfrac{1}{2}.$$

Hence, tne point on the line *A E* which shall mark the mean-proportional distance between *A C* and *A E*, must be to the left of *D*, somewhere near the letter *x*.

If we look for the reason why 3 is not a mean-proportional between 2 and 4, we find the cause, in what is called mathematical increment and decrement of magnitudes.

INCREMENT AND DECREMENT.

11. Natural law has so related numeric magnitudes of the cardinal numbers, that increased magnitudes decrease the rational proportion of the numbers, and the reverse. Thus: The rational proportion of 2 and 3 is $1\frac{1}{2}$ or $^3/_2$; the rational proportion of 3 and 4 is $1\frac{1}{3}$ or $^4/_3$; the rational proportion of 4 and 5 is $1\frac{1}{4}$ or $^5/_4$ and so on.

SCHOLIUM.

12. This apparently incongruous relationship of magnitudes as found in nature, is balanced by a law also provided by nature, which governs the adjustment of these differential proportions. This law is called the **Surd Law.** By proper application of the surd law the mean-proportional distance between any two linear values can be found and expressed in finite notation. But, as the surd law is part of applied commensurational arithmetic, which is not discussed in this section, the explanation of how to find the numeric mean-proportional value representing the mean-proportional distance between two given lengths, is reserved for the study of Section *C*.

Following, is a *geometric* demonstration which shows, by the exhibit 5, how a mean-proportional point is obtained, which marks a mean-proportional distance.

THE MEAN-PROPORTIONAL POINT.

13. The mean-proportional point is a point between two different linear magnitudes marked off on the same line from one common extremity.

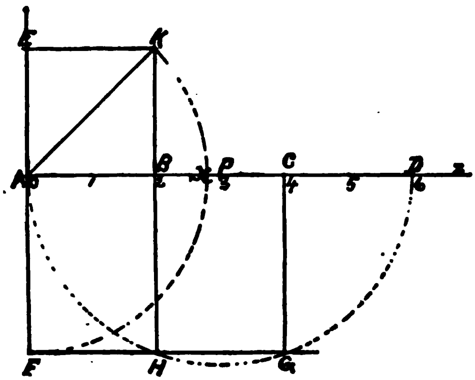


EXHIBIT 5.

A z is a given line of indefinite extent. *A* is the given common extremity. *A C* is a given line of the magnitude 4. *A B* is a given line of the magnitude 2.

In § 10, exhibit 4, it is shown, that 3 is not the mean-proportional point sought between 2 and 4. To obtain the true point, add the distance *A B* to *A C*, which gives a third line of the magnitude 6, marked in the diagram by the line *A D*. Now, if this line *A D* is made the diameter of a circle, the circle's radii are given as *P A* and *P D*. With the given radius *P A*, describe the semi-circle *A H G D*. Construct on the *least* of the two given lines (the line *A B*) a square perimeter as *A B K E A* and draw the diagonal *A K*. Extend the line *K B* to *H* on the semi-circle *A H G D*, and construct on the given rightangle *A B H* the rectangle *A B H F A*. Next, with *A K* as a radius, from *A* as a center, describe an arc from *K* to *F* which intersects the radius *A P* at a point to the left of the figure 3, and market *x*. This intersecting point, thus obtained, is tne true mean-proportional point sought between the lines *A B* and *A* C.

14. Now, as it is shown, by demonstration and construction, that *A F* equals *B H*, it is also shown that *A x* equals *B H*, since *A F* and *A x* are equal, as shown by the curve *K x F*.

Hence, the following axiom:

15. *The right sine of any arc in a given circle is a mean-proportional of the arc's versed sine and the circle's diameter minus the versed sine.*

This axiom is known as the **Canonic Axiom.**

Converted into a technical algebraic expression the "canonic axiom" may be expressed as follows:

$$\text{VS} : \text{RS} :: \text{RS} : Dmt - \text{VS}. *$$

That is: versed sine is to right sine as right sine is to diameter minus versed sine.

---

* The three sines: versed sine, right sine and co-sine, are abbreviated as above and expressed in small capitals, not in italics, thus : VS. RS. CS.

In the exhibit 5, x marks the mean-proportional point between the lines $A\ B$ and $A\ C$.

If, by substitution, we make $a=A\ B$, $b=A\ x$, $c=A\ C$, then we may say: $a : b :: b : c$.

And as $(A\ x=B\ H)=mp.$, and as $mp.=b$, we may also say:

$$a : mp. :: mp. : c.$$

In the diagram 5, five different lines are given, each of which equals the *mp.* If we ask to name them, they are represented by $A\ K$, $A\ F$, $B\ H$, $C\ G$ and $A\ x$.

CORRELATIVE AXIOM.

16. Any two lines of different, but definite magnitudes, are proportioned each to the other as is a given versed sine in a circle to two co-sines of the same arc of which the versed sine is given.

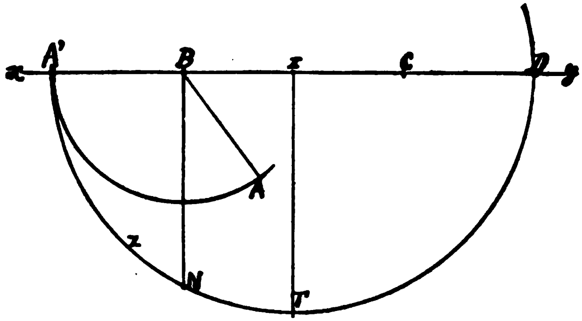


EXHIBIT 6.

DEMONSTRATION.

On any given line, as $x\ y$, lay off a distance as $B\ C$. From B mark another line as $B\ A$; lay off the given line $B\ A$ on $x\ y$ from B to A', and with compasses lay off the distance $B\ A'$ from C to D. Bisect $C\ D$ at I. From the center I, describe the semicircle $A'\ I\ D\ T\ A'$ and draw the bisecting radius $I\ T$. Finally, construct the right sine $B\ N$, then it is demonstrated, that the given minor line $B\ A$ equals the versed sine $A'\ B$ of the arc $A'\ z\ N$, while the major given line, $B\ C$, equals 2 co-sines of the arc $A'\ z\ N$. And it further follows, according to the canonic axiom, that $B\ N$ is the *mp* of $A\ B$ and $A\ C$.

If we substitute a for $A\ B$ and b for $B\ C$ the correlative axiom may be algebraically expressed as: $a : b ::$ vs. : 2 cs.

SCHOLIUM.

17. It must be understood, that the magnitude obtained by the process explained for finding the mean-proportional point and the consequent mean-proportional magnitude between two given linear magnitudes, not only relate to linear values, but to the areal values of sqares constructed on the lines, as well. For it is shown in exhibit 5, that the lines $A\,B$, $A\,x$, and $A\,C$ are proportioned as side :: diagonal :: 2 sides of any square. And it is shown §. 35–37, Part III., Section A, exhibits 17 and 18, that squares constructed on similarly related lines as $A\,B$, $A\,x$ and $A\,C$, are proportioned as 1 :: 2 :: 4.

CONVERSION.

18. Mathematical processes which change the figure of given geometric boundaries into a line, constitute *rectification;* a process which converts a given line into a boundary, is called a *transfiguration.* Any mathematical process which changes the relative position of any two given lines, boundaries, or planes constitutes *transposition.* Any mathematical process which changes the form of a geometric plane, constitutes *transformation.* Any mathematical process which changes the magnitude of a given geometric unit-measure, is called *fluxion.* Hence, there are five distinct mathematical processes by which changes are effected in geometric quantities; all of which come under the common name: **Conversion.**

RECTIFICATION.

19. Rectification means to straighten out given perimeters and given circumferences. Constructively, any perimeter can be rectified by the use of compasses and ruler, as the following exhibit shows.

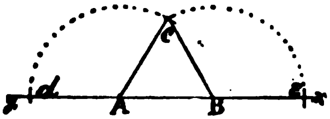


EXHIBIT 7.

DEMONSTRATION:

$A\ B\ C\ A$ represents the perimeter of a given trihedron. Extend the line $A\ B$ indefinitely to x and to y. Lay off with compasses, from A as a center, the line $A\ C$ to d, on $A\ y$; and likewise lay off, from B as a center, the line $B\ C$ to e on $B\ x$, then it is constructively shown, that the line $d\ e$ equals, in distance, the perimeter $A\ B\ C\ A$. That is to say, the line $d\,e$ represents the rectified perimeter of the given trihedron $A\ B\ C$. Perimeters of squares, polygons and other geometric forms, may similarly, and by the use of compasses and ruler be rectified into lines of equal extent.

20. By technical algebraic formula it may be said that $a=b$. But until the exact meaning of the two letters are given, the formula is worthless, since a and b represent different figures although their magnitudes (extent) are equal.

To give intelligent expression in this case to the equation $a=b$, say:

Perimeter $A\ B\ C\ A=a$

The line $d\,e=b$

$A\ B=1.$ $\overline{3\ A\ B}=A\ B\ C\ A.$ $a=mg.\ 3.$

$d\,e=3\ A\ B.$ $b=mg.\ 3.$

$a=b$ in *mg.*, but not in figure.

Since: $3 : a :: b : 3.$

21. Thus, it is shown, that *different geometric figures may be of one and the same magnitude.*

22. Circumferences can only be rectified by first finding some line definitely located in circles, which line equals, in extent, a known part of the circle's circumference. The line thus required

is one of the solving chords already described and defined in Section A, Part II, exhibit 14. The chord is known by name as the rectifying chord. This chord's peculiar relation to the circle's circumference is, that the chord equals, in extent, the curve which forms the circle's quadrant-arc, or one-fourth of the circle's circumference rectified into one straight length.

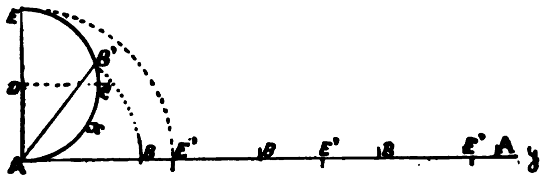


EXHIBIT 8.

DEMONSTRATION:

A E represents the diameter of a given circle. *D C* represents a radius which bisects the semi-circle, and the curve *A x C* represents the quadrant-arc or one-fourth of the circle's circumference. *A B* represents the rectifying chord subtending $^2/_7$th arc of circumference. Construct on *E A*, at *A*, the perpendicular line *A y*. Lay off the chord *A B* four times on the given line *A y*. Then, the distance *A A'* is obtained. That distance represents the rectified circumference of the given circle.

THE RECTIFIED PI-VALUE.

23. If instead of using the rectifying chord, the diameter of the given circle is used as a dividing measure for the rectified circumference, it is found by compasses, that the distance *A A'* contains three diameters plus the linear fraction *E' A'*. The fraction thus obtained is called the *pi*-fraction and the magnitude of the three diameters plus the magnitude of the *pi*-fraction constitute, as a whole, the so-called *pi*-value.

Thus, the *pi*-fraction is shown to be the *difference* of 4 rectifying chords and 3 diameters.

PART II.

TRANSFIGURATION.

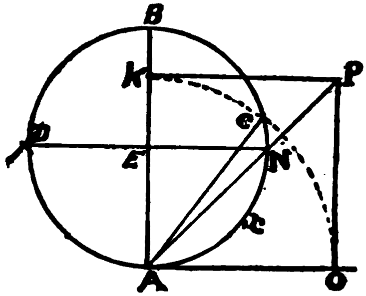


EXHIBIT 9.

SQUARING THE CIRCLE'S CIRCUMFERENCE.

24. Let the circle over the diameter $A\ B$ be quartered by the line $D\ N$. Draw the rectifying chord $A\ C$, and lay off that chord on the diameter $A\ B$, from A to K. Construct the line $K\ C$ perpendicular to $A\ B$, and construct the line $A\ O$ parallel and equal to $K\ P$. Next, join the points O, P, by the line $O\ P$ parallel to $A\ B$. Then, a square perimeter is given by the figure $A\ O\ P\ K\ A$, which perimeter equals in extent the circumference of the given circle over $A\ B$. The result is the same as that of folding up the rectified circumference (exhibit 8) into a square boundary. The process is called: *squaring a circle's circumference.* It must not be confounded with squaring the magnitude of the circumference.

25. Algebraically the equation may be expressed thus:

Substitute c for ○.

Substitute p for □.

Then, ○ = □ in *mg.*, but not in figure.

But, if c and p represent magnitudes of circumference and perimeter, then $c = p$.

Because: $c = \pi$ value.

$p = \pi$ value.

TRANSFORMATION.

26. Transformation means, to change a given plane or volume from one form to another.

In cases of transformation, the relation of boundaries to planes change. If a circle is changed to any other form, the extent of the boundary line increases; and it follows, if the circle's circumference is changed in its figure, as in the case of squaring the circumference (exhibit 9), the plane under this transfigured boundary is less than the plane under the boundary in circle-form. And it further follows: a given circle-plane transformed into a square, requires a longer boundary line than the length of its circumference.

Similar changes occur in the relation of planes and perimeters when angular planes are transformed.

TRANSPOSITION.

27. Transposition means, to change the relative position of two or more given geometric quantities; and as all transformation of planes involves transposition, the two processes are explained and illustrated together.

DEMONSTRATION.

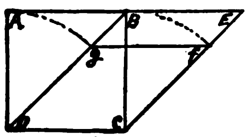


EXHIBIT 10.

28. *A C B D* is a given square composed of the two prime triangles *D B C* and *D A B*. Now, if the triangle *D A B* is transposed to the position *C B E*, then, the rhomboid *D E B C* is formed and composed of the same two prime triangles, as first given. Contemplating the change, it is found, that the boundary of the rhomboid is greater in extent than the boundary of the square, yet, the two planes given are of equal magnitudes. Closer analysis shows

that the *difference* of the two given perimeters' relative magnitudes can be obtained by laying off the line $D\,A$ on $D\,B$ at g, and by laying off the line $C\,B$ on $C\,E$ at f, for then it is shown, that the difference in the perimeters $A\,B\,C\,D\,A$ and $D\,B\,E\,C\,D$ is marked by the two lines $B\,g$ and $E\,f$.

29. The same differential relation between form, figure and magnitude presents itself here. For it could be said:

$$2\,(A\,B\,D) = D\,B\,C\,A \text{ and } 2\,(A\,B\,D) = D\,E\,B\,C.$$

And, if a is substituted for $2\,(A\,B\,D)$ in the square form, and b is substituted for $2\,(A\,B\,D)$ in the rhomboid form, then it does appear that the rhomb and the square are equal as regards magnitudes of plane; but they are by no means equal as regards form of plane or figure of perimeters. Hence, when an algebraic equation is presented as: $a=b$, it does not follow that anybody knows what it means, until it is explained.

These examples of algebraic expressions relating to converted forms and figures, are presented for the purpose of showing the lose manner in which "algebra" can be applied in mathematics and is applied in "calculus."

TRANSPOSITION OF LINEAR ELEMENTS.

30. Transposition, also means, to change by compasses or ruler, linear elements within a given circle, and it follows, that one given plane of a certain form within another giving plane, can be transposed by compasses and ruler from one position to another within a greater plane.

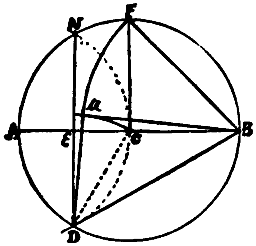


EXHIBIT II.

DEMONSTRATION, 1.

31. In the given circle over the diameter *A B* with *c* for a center, describe the line *e F* which bisects the semicircle *A B F A*. Draw the quadrant chord *B F*. Draw the tridrant chord *B D*; draw another tridrant chord *D N*, which bisects the radius *c A* at the point *e*, then, the prime triangle *B c F* is given.

Let the quadrant chord *B F* (which is the hypotenuse of the given triangle) be the line given for transposition, so that, instead of posing as the hypotenuse of a rightangled triangle, the line shall pose as a rightangled triangle's leg. To do that, make *D* a center and *D e* a radius, with which describe the curve *c u*. Likewise, make *B* a center and *B F* a radius, with which describe the curve *F u*. These two curves mark the intersecting point *u*. Next, draw the lines *D u* and *B u*, which, with the tridrant chord *B D* form another rightangled triangle (*B u D B*) of which, the line *B F*, obtained by the process of transposition, has become the major leg *u B*.

DEMONSTRATION, 2.

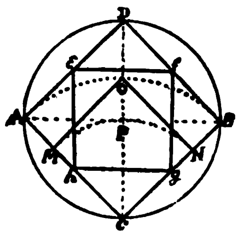


EXHIBIT 12.

32. *A B D C* is a given square inscribed in a circle, and *e g h f* is a minor square inscribed in the given major square *A B D C*. In the relative positions of these two squares it is easily proven that the planes of these two squares are proportioned in magnitudes as 1 : 2, since, the four prime triangles bounding the minor square can be shown by transposition to equal four similar prime triangles composing the minor square, as shown by drawing

diagonals from *f* to *h* and from *e g*. But, if the question is to find the difference of the side of the minor square's perimeter and the side of the major square's perimeter, then, the minor square, must as it were, be *turned* round, so that the two sides of the different squares can be compared. That is to say: the minor square must be placed in a parallel position to the major square so that the minor side becomes part of the major side.

This is done by compasses and ruler as follows:

From *C* as a center with radius *C P* describe the quadrant *C M P N C*. From *C* as a center with radius *C B* or *C A* describe another quadrant as *C B O B C*, then, the point *O* is given. Draw lines from *N* to *O* and from *M* to *O*. Now it is found that the square *M N O C* equals the square *e g f h*, and consequently, the lateral difference of the two squares is shown to be the distance *M A* or *N B*. In other words, by the process described, the first given inscribed square, has, as it were, been turned round, so as to occupy a parallel position to the circumscribed square.

SQUARING THE CIRCLE.*

33. Squaring the circle means: to find a square plane of equal magnitude to the plane of a given circle.

The demonstration involves all the given processes of conversion. The principal factors employed in the squaring process are the rectifying and the squaring chords; and the equal square to circle is found to be a mean-proportional between the square constructed on the circle's diameter and another square of equal perimeter to circle's circumference.

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* The two greatest mathematical problems, on account of the importance to science the solutions thereof involve, are, the "squaring of the circle" and the "cubing of the sphere". The cubing of the sphere can not be satisfactorily explained or demonstrated without the aid of commensurational arithmetic, hence that demonstration and solution is left to section C. The demonstration of how to square the circle, constructively, with compasses and ruler, is given in the following exhibit.

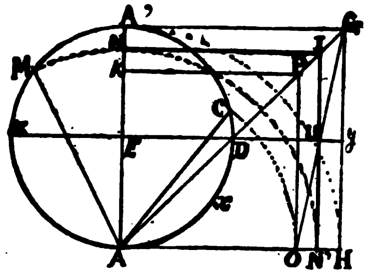


EXHIBIT 13.

DEMONSTRATION.

34. Given: a circle over the diameter $A\,A'$; E is the center and $E\,A$ the radius.

$A\,C$ is the rectifying chord equal to the quadrant arc $A\,x\,D$.

With $A\,C$ construct the square perimeter $A\,O\,P\,K$ equal to circumference (in extent).

Construct on diameter $A\,A'$ the square $A\,H\,G\,A'$. Draw the line $G\,O$ and construct the perpendicular $E\,y$ parallel to $A\,H$. The intersecting point U, marks the mean-proportional distance between $A\,O^2$ and $A\,H^2$. A line drawn perpendicular to $O\,H$ and parallel to $G\,H$, through the point U and terminating at I and N', represents the side of equal square to circle, which square also is the mean-proportional of the square on diameter and the square of equal perimeter to the circle's circumference. That is:

$$A\,O^2 : A\,N'^2 :: A\,N'^2 : A\,H^2$$

By substitution, make $a^2 = A\,O^2$,

$mp.^2 = A\,N'^2$, make $\blacksquare = \odot$,

$b^2 = A\,H^2$.

Then, $a^2 : mp.^2 :: mp.^2 : b^2$.

Or, $a^2 : \blacksquare :: \odot : b^2$.

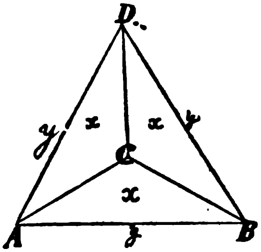
35. Now, when the side of $mp.^2$ is laid off in the circle from A to M as a chord, it is found that the chord $A\,M$ is the squaring chord as described, in Section A, Part II., § 27, exhibit 15.

FLUXION.

36. Fluxion is a mathematical process through which surd* unit-measures are changed into solvent** unit-measures, by changing the form or the magnitude of the units. The process is based on the fact that nature has provided every definite geometric form with a fitting unit-measure which in some way is commensurately related to the boundary of the form. But, while nature provides such unit-measures, it is left for the mathematician to find them. When found, these natural measures are called "solvents."

37. Geometrically, the trihedron and the square illustrate plainly the difference of surd and solvent unit-measures.

FIG. 1.



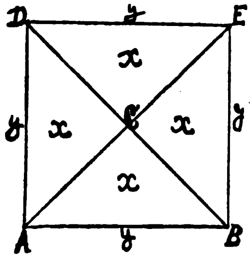


FIG. 2.

EXHIBIT 14.

Fig. 1 shows the trihedron perimeter $A\,B\,D\,A$, which exhibits the three linear units $A\,B$, $B\,D$ and $D\,A$; each one of which are symbolized by the letter y. If lines from the center c are drawn to the points D, A, B, three areal units, in the form of acute-angled isosceles triangles are given, each of which is an aliquot divisor for the given trihedron's plane. These natural areal units are symbolized by x.

---

* A surd unit-measure is one, which, on account of form or magnitude, is *not* an aliquot divisor for a given plane.

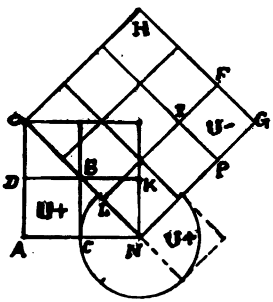
** A "solvent" is a unit-measure obtained through conversion of a surd unit-measure, either by changing the form or increasing or decreasing its magnitude, sc it can be used as an aliquot divisor.

Fig. 2 shows the square perimeter $A\ B\ E\ D\ A$, which exhibits the four linear units $A\ B$, $B\ E$, $E\ D$, $D\ A$, each one of which is symbolized by the letter y. If lines from the center c are drawn to the points A, B, E, D, four areal units in the form of prime triangles are given, each of which is an aliquot divisor for the given square's plane. These natural areal units are symbolized by x.

38. Now it is evident and can be proven by trial, that the natural unit-measure of the trihedron is not a fitting aliquot divisor for the square plane; neither is the natural unit-measure of the square a fitting aliquot divisor for the plane of the trihedron.

39. Aside, however, from this geometric aspect of fluxion, depending on form for fitness, the arithmetical aspect of magnitude, is of even greater importance, as on it depends the economic processes by which correct results are obtained in mathematical computation.

To illustrate the application of fluxions to computation by way of fitting magnitudes, let the following exhibit serve as explanation.



EHHIBIT 15.

40. On a given line as $A\ C\ N$, composed of the two linear units $A\ C$ and $C\ N$, construct a square composed of four areal units, one of which is marked $A\ B\ C\ D$. On the diagonal $O\ N$ of

the first given square, construct another square, and divide its plane into nine areal units, one of which is marked $P F G I$.

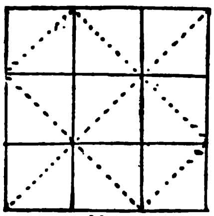
On trial with compasses (making N a center and $N C$ a radius) it is easily shown, that the magnitude of the unit (four of which compose the $A N^2$) is greater than the magnitude of the unit, nine of which compose the $O N^2$. To mark the distinction of these two unit-measures (equal in form, but different in magnitude) the major unit $A C^2$ is symbolized $U +$, while the minor unit PG^2 is symbolized $U -$.

41. Now, the mathematician's business is to find the true difference in the magnitudes of these two natural units.

With care, the true relation of these two units is obtainable by the use of compasses and ruler in the following manner.

On a large scale construct the diagram given in exhibit 15. Divide one of the *major* units into nine equal square parts. Divide one of the *minor* units into eight equal prime triangles, thus:

FIG. 1.



U+

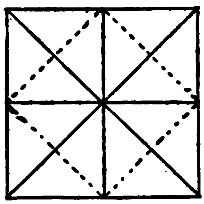


FIG. 2.

U−

EXHIBIT 16.

Then it is found that, when each of the nine square parts (Fig. 1) are made into two prime triangles, and each of the eight prime triangles (Fig. 2) are made into two prime triangles, then, $U +$ contains 18 prime triangles and $U -$ contains 16 prime triangles, all of which triangles are equal, both in form and in magnitude. The same result is also obtainable by converting the

16 triangles of $U-$ into 8 squares and by converting the 18 triangles of $U+$ into 9 squares, for then it is shown, that the difference in the magnitude of $U-$ and $U+$ is just *one* of the square parts into which $U+$ is divided when first given.

Here, it must be remembered, that each of the parts composing the magnitude of a unit* is technically called a nit. Hence, the difference of $U+$ and $U-$ may be expressed, thus:

$$(mg.\ U+) = 9\ nts.$$
$$(mg.\ U-) = 8\ nts.^{**}$$

DOUBLING THE SQUARE.

42. The difficulty of doubling the square has been, that every integral square, the magnitude of which is given in unit-value, when doubled, produces a surd value not divisible by the given unit of the integral square, without leaving remaining fractions of unknown value. This difficulty is overcome by "fluxions"; for, by the use of two units of different magnitudes, suited to the computation, the doubling of the square becomes a simple operation, thus:

FIG. 1.

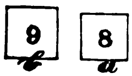


EXHIBIT 17.

a and *b* represents two square units of the respective magnitudes 8 and 9. (Fig. 1.)

FIG. 2.

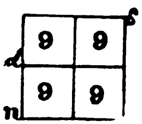


EXHIBIT 18.

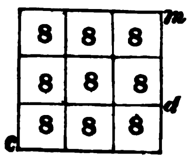


FIG. 3.

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* See Section B, Part I, Exhibit 3.

** Later on, in Section C, it will be shown that the magnitude 8 is the least solving value which occurs in geometric computation.

Compose a square root of $3a$, as shown by $c\ d$, fig. 3. Compose another square root of $2b$, as shown by the root $d\ e$, fig. 1. Square the root $d\ e$ and produce the square plane $n\ e$ of the magnitude 4 units. Square the root $c\ d$ and produce the square plane $c\ m$ of the magnitude 9 units. Now, although the number of units contained in each square are not in the required proportion to prove that the area of $c\ m$ is double the area of $n\ e$, it is found, when both these squares are converted into equivalent nit values, the required proportion is obtained, and the doubling of $d\ e^2$ is proven by $c\ d^2$. For it is self-evident that $9a$ are twice the value of $4b$, since $9 \times 8 = 72$,

and $9 \times 4 = 36$,

and $2 \times 36 = 72$.

43. Thus it is shown, that the common conventional ruling: 2 and 2 make 4, has exceptions; since it is demonstrated, that

$$(4b + 9a) = 12b.$$

That is: $(\overline{4b=36} + \overline{9a=72}) = 108$,

and $\overline{108 \div 9} = 12b$.

By substitution, the relation of the two squares $e\ n$ and $c\ m$, may be expressed in the following equational formulas:

Substitute a for the square root $d\ e$.

Substitute b for the square root $c\ d$.

Then, $a^2 : b^2 :: (2\ U+)^2 : (3\ U-)^2$,

and $\sqrt{4\ U+} : \sqrt{9\ U-} :: a : b$.

RELATION OF SIDES AND DIAGONALS OF SQUARES.

44. The foregoing demonstrations show how squares may be doubled by areal unit-measures of diverse magnitudes, independently of the lateral values which express the relation of the

squares' perimetric extent, a relation which can not be determined by any other than a common measure.

The first of these perimetric relations is that of two square perimeters the planes of which are proportioned as 1 : 2 (see exhibit 15). And as the sides of all squares so proportioned are related to each other as are sides and diagonal of squares (see exhibit 15) it follows, that the difference of the lateral magnitudes of the two given areal unit-measures must be found before commensurational relationship can be established between proportions of perimeters and proportions of planes. The principle which underlies this relationship is found in the natural correlative increment and decrement of square planes and square perimeters in exact harmony with increased and decreased magnitudes of either plane or perimeter.

45. To illustrate the full meaning of these statements, let *A C B D* be a given square of the magnitude 25 units.

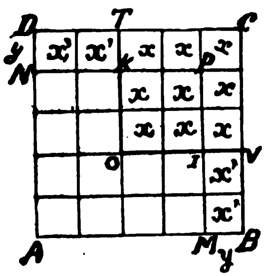


EXHIBIT 19.

Let a portion of the area, as *O T C V O*, be taken away, and let us ask the question: how much shall be taken away from the perimeter, so as to form another square of the remaining portion: *D T O V B A D*? The portion to be taken away from the plane as originally given is marked by nine x, each of which represents

one areal unit. Now, transpose 4 of these units (x), contained in the space $P\,O\,K\,I$, to the spaces $T\,K\,N\,D$ and $I\,V\,M\,B$, and transpose the 4 x' to the place left vacant in $K\,I\,P\,O$. Then it is seen, that the portion $T\,V\,O\,C$, originally given in a square form, has been transformed into the *elbow*-form $N\,D\,C\,B\,M\,P\,N$. For it is evident, that the area contained under either of these boundaries are equal to nine of the the originally given areal units. And now it is shown, through this transformation, that the corresponding reduction of the perimetric extent to that of the areal reduction of the plane, is marked by twice the distance $M\,B$ and twice the distance $D\,N$. In other words it is found, that the areal differenc of $A\,M^2$ and $B\,A^2$ is 9 areal units, while the linear difference of the side $A\,M$ and $A\,B$ is one linear unit.

46. Taking this as a common principle for all natural increment and decrement of any square's perimeter and plane, let any definite portion taken from a given square be what it will, a corresponding portion of the given square's perimeter can be found by converting the areal portion to be subtracted, into the *elbow*-form. Likewise, let any definite areal portion be added to a given square, a corresponding portion to be added to the given square's perimeter, can be obtained by similar process of transformation.

47. Henceforth in operations called fluxions, the technical expression for any part added to or taken from a square plane, is designated by the letter x. Any part added to or taken from a square perimeter's side, is designated by the letter y.

Converted into numeric values, and treated arithmetically, these quantities are respectively named, the x value and the y value.

48. From these first principles of fluxions are deducted a series of formulas on which the Surd Law is founded.

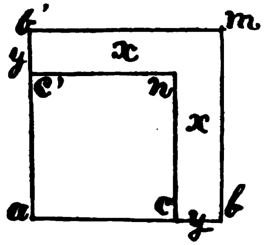


EXHIBIT 20.

Exhibit 20 shows, first, a minor square $a\ c'\ n\ c$, to which is added a certain area in elbow-form marked x. But the same exhibit shows also a major square $a\ b\ b'\ m$, from which is taken a certain area in elbow-form marked x. The corresponding linear fractions marked y applies alike to either square with the difference, that, when the minor square is augmented in area, the corresponding fraction is designated $+ y$, while, when the major square is reduced in area the corresponding lateral fraction is designated $- y$.

49. Algebraically, we may say:

	$a\ c = a.$	$a\ b = b.$
Then:	$a + y = b.$	$b - y = a.$
	$(a + y)^2 = b^2.$	$(b - y)^2 = a^2.$
	$a^2 = (b^2 - x).$	$(b^2 = (a^2 + x).$
	$.\sqrt{b^2} = a + y.$	$.\sqrt{a^2} = b - y.$

Arithmetically, if we call $a^2 = 1$.

Then: $\overline{1 + y} = b$, and $b^2 = 1 + x$.

Hence, $(1 + y)^2 = 1 + x$.

And, $.\sqrt{1 + x} = 1 + y$.

And, $(1 - y)^2 = 1 - x$.

And, $.\sqrt{1 - x} = 1 - y$.

SUMMARY OF PART II, SECTION B.

TECHNICAL ALGEBRA:

Mean-Proportional Values—Proportional Points—Mean-Proportional Distances—Geometric Increment and Decrement—The Canonic Axiom and its Correlatives.

CONVERSION:

Rectification — Transfiguration — Transposition — Transformation—Fluxion.

FUNDAMENTAL PRINCIPLES OF FLUXION.

SUPPLEMENT.

SUGGESTION TO THE TUTOR.

The importance of having the difference of the old and the new mathematical methods pointed out by the tutor to students, need scarcely be alluded to, as every intelligent teacher will, no doubt, appreciate the convenient manner in which the subject is treated in the preface. Whatever is new, is briefly embodied in the twenty eight findings and discoveries. And it would be good exercise for the pupil to study the various points of difference, pointed out by the tutor.

SECTION B

PART III

SECTION B.

Part III.

EVOLUTION OF MAGNITUDES.

1. The principle of perpetual evolution of squares proportioned in magnitudes as 1, 2, 3, 4, 5, 6, etc., is illustrated by using the diagonal of a square for the generator of a prime rectangle, and by using the diagonal of the prime rectangle for the generator of the cube rectangle, and so on, always use the diagonal of the last-formed rectangle as a generating factor for the production of the major side of the succeeding rectangle. See Exhibit 11, § 22, Section A, Part III.

PERPETUAL SCALE OF SQUARE MAGNITUDES.

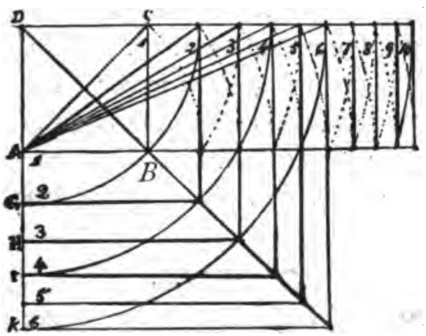


EXHIBIT I.

2. *A C D B* is a given square unit-measure.

From this given unit-measure is raised a series of roots and squares as already explained. The magnitudes of these squares are in the ratio of the cardinal numbers 1, 2, 3, 4, 5, 6, etc. That

is, DG^2 represents the *dual* square, a square of *mg.* 2; DH^2 represents the *tri* square, a square of *mg.* 3; DI^2 represents the *prime* square, a square of *mg.* 4, which square is the *least* integral square.

Constructively it is shown, that the square *mg.* 6, is the dual square of *mg.* 3, and likewise, the prime square *mg.* 4, is the dual square of *mg.* 2.

Thus, an endless series of squares in cardinal proportions can be developed and evolved by continuing the process. Apparently, the operation may result in so small a difference of the *Gf.* and the side of the rectangle which represents the root, that in the mechanical construction no difference is visible to the eye, yet, the operation can still be proceeded with by making the dual square (*mg.* 2) a unit-measure and evolve as before until the series again seems exhausted for want of visual capacity of the operator to discern the difference which he, nevertheless, knows is there.

3. In the exhibit, the *Gf.* is represented by the several diagonals *A* 1, *A* 2, *A* 3, *A* 4, and so on. Analytically examined it is found, that the generating factor is always the hypotenuse of a rightangled triangle, the minor leg of which is the minor side of a given root, while the major leg is the major side of the same root.

And since the constant increase in the magnitude of the squares is *one*, the fact, *that the sum of the squares on any rightangled triangle's legs equals the square on the triangle's hypotenuse*, is clearly demonstrated.

SCHOLIUM.

4. The discovery of the generating factor through which square magnitudes are evolved, is one of the most important discoveries in science. Pythagoras, the Greek geometer, demonstrated that the squares on the legs of any rightangled triangle, taken together, equal the area of a square, constructed on the hypote-

nuse of the triangle. But his demonstrations only applied to triangles, the legs and hypotenuse of which were so related in their magnitudes that they constituted sides of integral squares. Now, since a rule has been found by which all square magnitudes can be found and related together in commensurate order; and as by conversion every geometric figure and form is convertible into a square, it follows, that the exact relative values of any two given forms converted into squares are obtainable, if the magnitude of their respective solvents can be found.

SQUARE'S RELATION TO CIRCLES.

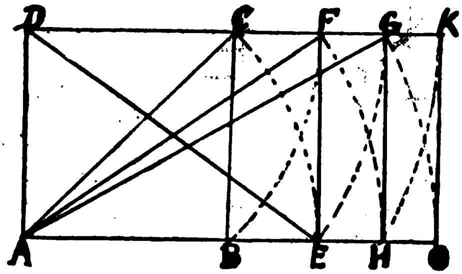


EXHIBIT 2.

5. Exhibit 2 represents the prime square root or the root of the least integral square (*mg.* 4). *A C D B* is the basic unit and the prime surd value. *A F E D* is the surd root of the dual square. *A G H D* is the surd root of the tri square. Squares constructed on *A B*, *A E*, *A H* are all surd squares. Now, it is found, that the three generating factors *A C*, *A F*, *A G* are proportioned as are the quadrant chord, the tridrant chord and the diameter in any circle. And it follows, that a circle described with the radius *A B* over the diameter *A O*, is commensuaratelyrelated to this first series of evolved squares, which suggests the correlative evolution of circles' magnitudes by similar process as that employed in the evolution of squares. Hence, it may be said:

Circles are to each other as the squares on their diameters.

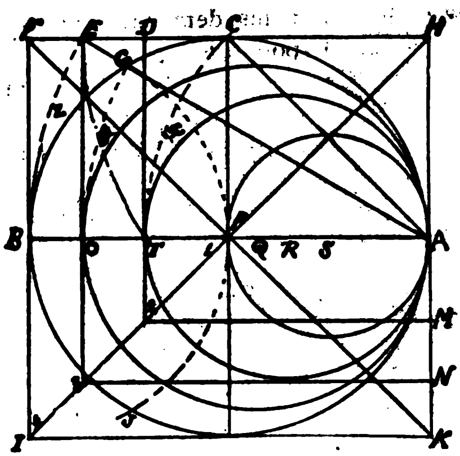


EXHIBIT 3.

6. To illustrate the evolution of circles, construct on a given line as $A\ B$ the unit $A\ C\ H\ P$, the dual square root $A\ D\ T\ H$, the tri square root $A\ E\ O\ H$ and the prime square root $A\ F\ B\ H$. With $P\ A$ for radius, describe the great circle over $A\ B$. Quarter the great circle and draw the quadrant chord $A\ C$; draw the tridrant chord $A\ G$ and extend the same to E. Then it is found, that the quadrant chord $A\ C = A\ T$ and the tridrant chord $A\ G = A\ O$. Now, bisect $A\ P$ at S, bisect $A\ T$ at R, bisect $A\ O$ at Q, and from the points S, R, Q as centers, describe the several circles over $A\ P$, $A\ T$, $A\ O$. Finally mark the squares 1, 2, 3, 4, then it is shown, that the several circles evolved are proportioned in magnitudes as the numbers 1, 2, 3, 4. And it is also shown: that a circle over any given circle's radius is proportioned to the given circle as 1 : 4; that a circle over any given circle's quadrant chord is proportioned to the given circle as 1 : 2; that a circle over any given circle's tridrant chord is proportioned to the given circle as 3 : 4.

EVOLUTION OF CUBES.

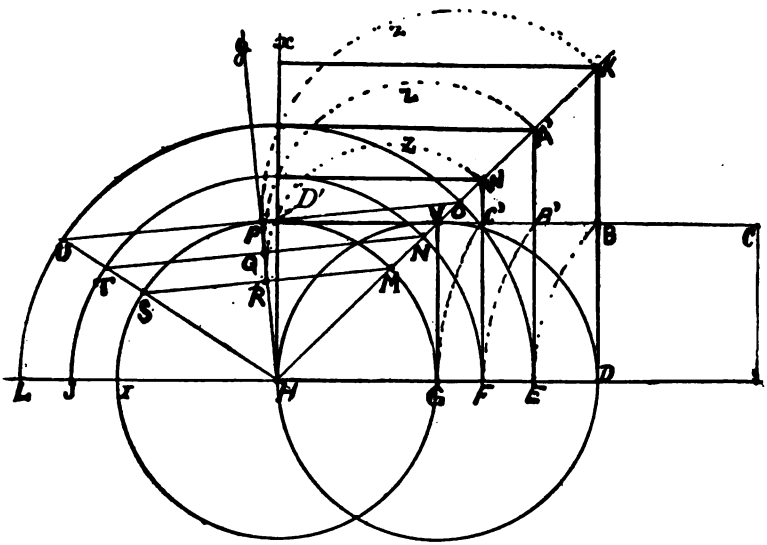


EXHIBIT 4.

7. *A B C D* is a given square representing the face of a cube unit. On this given square construct a prime rectangle as *A V C G* comprising the root of a dual square, the root of a tri square and a prime square root; then it is found, if squares are raised over these roots, the squares represent faces of cubes proportioned in volumes as 8, 4, 2, 1. To illustrate this fact, transpose the square *A B C D* to the position *H V G D';* make *G* the center of a circle described over the diameter *H D*, and construct the squares HF^2, HE^2 and HD^2. Then the squares HG^2, HF^2, HE^2, HD^2 represent the faces of cubes proportioned in volumes as 1, 2, 4, 8. Hence, it may be said:

Cubes are doubled in inverse ratio of evolved squares.

8. Analytically examined, it is found: When a quadrant is inscribed on the face of any given cube and the octant arc of the quadrant is rectified into a line and added to the diagonal of the given cube's face, the diagonal of a greater cube's face is given, the volume of which cube is double that of the cube first given. So on, by repeating the process a continuous series of doubled cubes can be evolved.

9. To illustrate the process, let *G D' V H* be the face of a given cube. Draw the diagonal *H V* and extend the same indefinitely beyond the point *K*. From *H* as a center, describe with *H G* a circle over the diameter *G I*. Then the quadrant *H G M D' H* is inscribed on the given cube's face (*H V D' G*). From the point *M*, lay off $^2{}_7$th of the circle's circumference over *G I* (at the point *S*). Then draw the chord *M S* which is the rectifying chord of the circle over *G I*. And as the rectifying chord equals in extent the circle's quadrant arc, one-half the chord equals the rectified octant arc required for the extension of the diagonal *H V*. Draw the line *H S* which completes the construction of the sector *H M P S*. When this sector is bisected by the line *H P* and extended to *y*, the rectified octant arc is given in the line *M R*. Now, make *M* the center and *M R* a radius, and describe from *M* with *M R* the curve *R z W*, intersecting *H K* at *W*. Then it is found, when a line from the point *W* is drawn parallel to *V G* and perpendicular to *H D*, that line marks the side of the tri square evolved from the unit *A B C D*. Now extend the line *H S* to beyond *U*, and proceed as before to inscribe quadrants on each successively evolved cube face; then draw as many rectifying chords as required, and from the bisecting points on these several chords, draw the curves *Q z A'* and *P z K*, which mark on the line *H K*, the increased magnitudes of the cubes' faces.

APPLICATION OF THE GENERATING FACTOR TO GEOMETRIC CONSTRUCTION.

EXAMPLE 1.

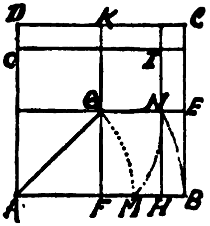


EXHIBIT 5.

10. Let it be required to mark within a given prime square the x value of 2^2-1. Or, which is the same thing: mark the boundary of the tri square within a given prime square. On the root of the prime square *A E* mark with the *Gf.* the points *M* and *N* and *H*. Then, the distance *E N* is the y value $= H B$, and *A H* is the side of the tri square required, which, when constructed, as *A H I O A*, marks the x value (in elbow-form) equal to one unit. Hence, $A H^2$ represents 2^2-1.

EXAMPLE 2.

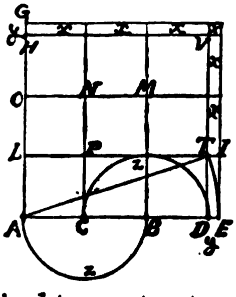
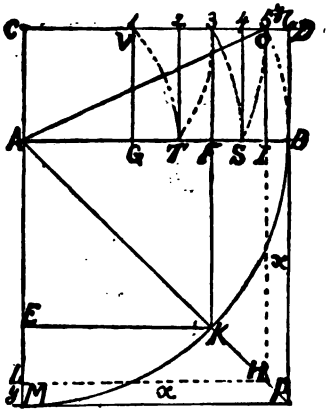


EXHIBIT 6.

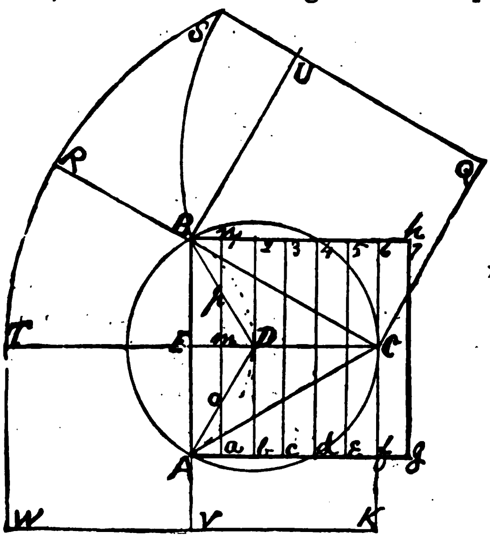
11. Let it be required to construct a square composed of the prime square and all the surd values numbered from 1 to the prime square. *A M B O* is the given prime square; *V B* represents the surd value 3; *M H* represents the surd value 2; the elbow-form x represents the surd value 1. Hence, the sum of all the surd values below the prime square, plus the prime square, is the least surd square between 3^2 and 4^2, and its magnitude is 10 as shown by the generating factor *A T*.



EXAMPLE 3. EXHIBIT 7.

12. Let it be required to rectify the superfacies of a given cube into one square form.

A V G C represents the face of the cube and six such planes represent the superfacies of the cube. Construct the rectangle *A D B C* which represents the surd root of the square *mg.* 6. On *A B* construct the square *B M P A* which equals in area the superfacies of the cube, one face of which is given as the square *A V G C.*



EXAMPLE 4. EXHIBIT 8.

PART III.

13. Application of the *Gf.* shows that twice the altitude ^{2} equals trice the square on the trihedron's side. For, let $A\ B\ C$ represent a given trihedron, $E\ C$ represents the altitude and $C\ B$ represents one side; then we find, by doubling the distance $C\ E$, the side of a prime square is given as $C\ T$. And when a square is constructed on the side $B\ C$ and is augmented to a tri square, as shown by the curves $B\ S$ and $S\ R$, then it is found, that $C\ R^2 = C\ T^2$, yet, one square is of the *mg.* 3 and the other, of the *mg.* 4. This discrepancy is explained by the difference in the magnitudes of the units which compose the areas. For it is selfevident that the square on the altitude of the trihedron is less in magnitude than the square on the trihedron's side, since three of the latter equal four of the former. And since the three squares on the trihedron's perimeter equal four squares of equal area to the square on the trihedron's altitude, it follows, that the square on any trihedron's altitude is proportioned to the sum of the squares on the trihedron's perimeter as 1 : 4.

TRIGONOMETRY.

14. Trigonometry is that part of geometry which correlates together angles, arcs and sines.

This correlation of lines and curves involves exact and accurate division of diameter and circumference of the circle, so that, any one part of the circumference is measurable by a definite part of the radius.

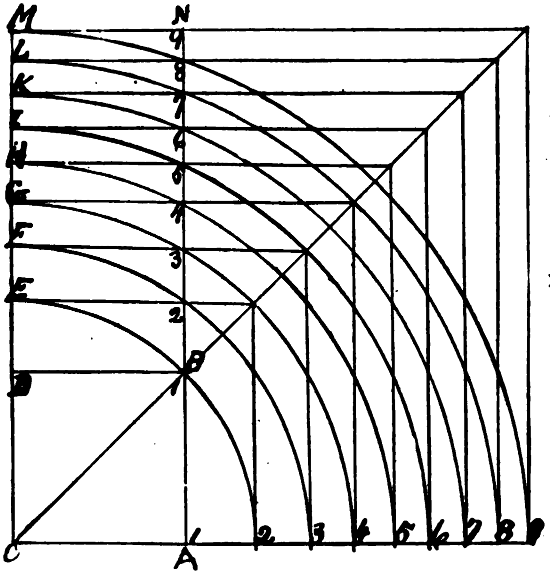


EXHIBIT 9.

15. Exhibit 9 shows a part of what is called a trigonometrical canon of proportion. In principle, the scale is founded on a series of evolved squares the lateral values of which, reduced to finite notation, represent the natural commensurational division of the sines and the arcs, so that, when a right sine is given the versed sine and the co-sine corresponding is also given. Likewise, when an angle is given, the corresponding arc is given. As all triangles comprise six geometric elements, viz: three angle and three sides, any three of these six values given, determine the other three; and in as much as the perimeter of any triangle is composed of chords and sines, trigonometry constitutes a fundamental method applicable alike to mensuration, navigation and astronomy.

THE GEOMETRIC CANON OF PROPORTION.

16. While no limit can be set to possible magnitudes of geometric elements, nature has provided a limit to commensurational

division by subdivision of the true pi value into a number of nits and finits (obtained by evolution of squares) which finally aggregate in a sum terminating in ciphers.

Calculation beyond this is carried on by multiples of tens which will increase the commensurational values to any required extent without changing the inherent proportions of the geometric elements.

When this terminal commensurational sum is divided up among a sufficient number of sines, arcs and angles, numerically expressed, and properly grouped and tabulated for convenient and economic use in computation, as substitutes for the so-called irrational logarithms, such a formulated instrument of calculation is called the Geometric Canon of Proportion.

EVOLUTION OF TRIHEDRONS.

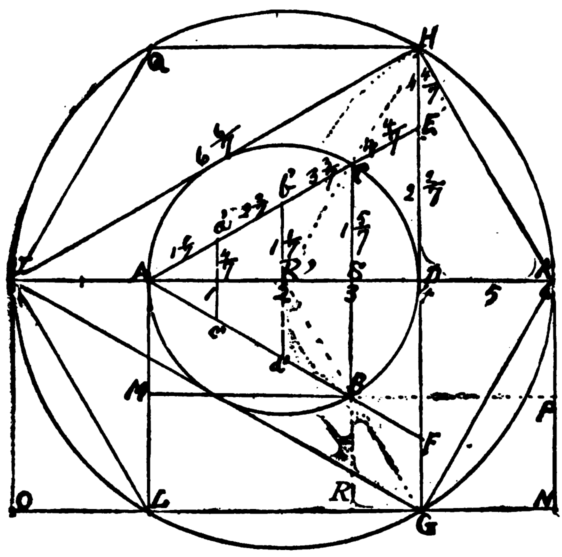


EXHIBIT 10.

17. When a series of trihedrons are constructed on a given line, as $A\ D$ divided into four equal parts, each of which parts represents the altitude of a given trihedron and all of which altitudes are proportioned as the cardinal numbers 1, 2, 3, 4, etc., then it is found, that the sides and altitudes of these several trihedrons are proportioned by ratio, in their magnitudes, as the numbers 8 and 7.

18. When a trihedron is inscribed in a circle, as the triangle $A\ B\ C\ A$, it is found, that the circumscribed circle over the diameter $A\ D$ is an inscribed circle in another trihedron, as the triangle $I\ G\ H\ I$. This great triangle is found to be proportioned in area to the area of the triangle $A\ B\ C\ A$ as 4:1. Hence, it may be said:

An inscribed trihedron is proportioned to the circumscribed trihedron as 1 : 4, which is shown in the following exhibit:

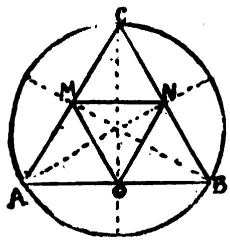


EXHIBIT II

19. And it is shown by the exhibit 10, that an inscribed trihedron and an inscribed hexagon in the same circle are proportioned in areas, as 1 : 2. For it is evident to the eye, that the two triangles $I\ Q\ H$, $H\ K\ G$, $G\ L\ I$ can be formed into a trihedron equal to the triangle $I\ H\ G\ I$.

20. By conversions it can be shown that the triangle $A\ B\ C$ is equal in area to one-fourth of the triangle $I\ G\ H$. It is done by transposing the area $A\ S\ C$ to $A\ M\ B$, whereby the rectangle $S\ M\ A\ B$ is produced. This rectangle equals the triangle $A\ B\ C$, since it contains the two triangles $A\ S\ C$ and $A\ S\ B$. And the

diagram shows, that the rectangle *S M A B* also equals one-fourth the area of the greater rectangle *I G D O*, which is composed of the two triangles *I G D* and *I D H*, which, together, compose the great triangle *I G H I*. To fully illustrate this fact, transpose the area *A O I L* to *D N K G*, for then, when the line *M B* is extented to *P* and the line *S B* is extended to *R*, it is demonstrable by compasses and ruler, that *A B S M* equals one-fourth of the area *A N K L* which is equal to the great trihedron *I H G I* transformed into a rectangle.

21. Thus it is shown, that commensurational relation exists between trihedrons and circles, as well as between squares and circles. The characteristic difference in the relation of these forms, is this: inscribed and circumscribed squares in and about the same circle are proportioned as 1 : 2, as also are inscribed and circumscribed circles in and about the same square; but, inscribed and circumscribed trihedrons in and about the same circle are proportioned as 1 : 4, as also are inscribed and circumscribed circles in and about the same trihedron.

THE SOLVING TRIANGLES.

EXHIBIT 12.

22. The solving triangles are three right-angled triangles having a common hypotemuse (the diameter of a circle) and differently proportioned legs. Exhibit 12 shows the circle, its diameter and the three solving chords already described in Section A, Part II, page 11, § 22–30. These chords, viz.: the rectifying chord, the squaring chord and the cube chord occupy positions relative to the diameter as major legs of right-angled triangles. That is to say: if chords are drawn from $^2/_7$ to 6, from $^6/_{17}$ to 6, and from extreme of the cube chord to 0, then, these three solving triangles are called after the chords: the rectifying triangle, the squaring triangle and the cube triangle, and when these triangles are cut out separately, they can be used as protractors for marking in any circle the three solving chords.

THE SOLVING TRIANGLES.

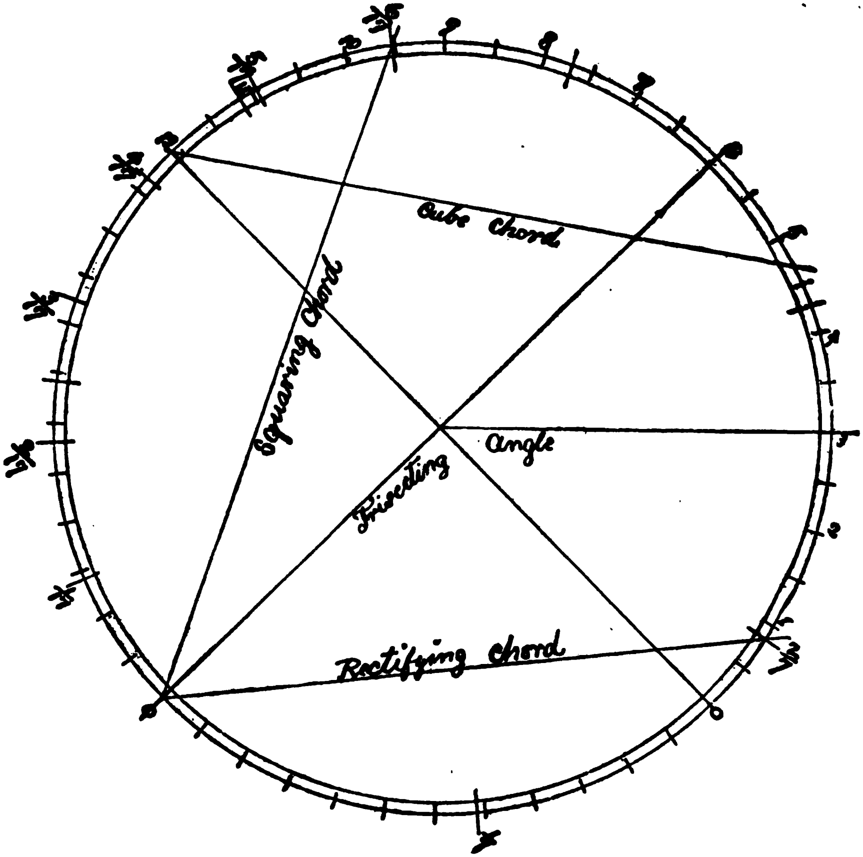


EXHIBIT 12.

The identifying characteristics of these triangles consists in the squares on their perimeters and in the angles of their planes formed by the perimeters.

SCHOLIUM.

23. It must be understood that the mutual relation of squares on triangular perimeters differ with the *class* to which the triangles belong. Only rightangled triangles have the squares on their perimeters so related, that the two squares on the legs, equal the one square on the hypotenuse; and as this unerring and con-

stant relation is of great economic value to the student of trigonometry, every other kind of triangle should be divided into rightangled triangles and thus computed. It must also be understood that in every triangle which has one rightangle, the sum of all the angles contained in the plane equal two rightangles. Hence, when two of the angles in any rightangled triangle are given, the third is also given, since the third is the difference of the given rightangle plus one given acute angle, or else, the third is the sum of the two given acute angles. For it is evident, that in every rightangled triangle there must be two acute angles and one rightangle; and it follows, that the acute angles determine what the rightangle shall be numerically denominated. The rightangle, it is true, always measures one quadrant arc of the circle, but it is not true, that the quadrant arc can be arbitrarily divided into a fixed number of parts which give the same numeric value to every rightangle. The following illustrations explain these remarks.

THE CUBE TRIANGLE.

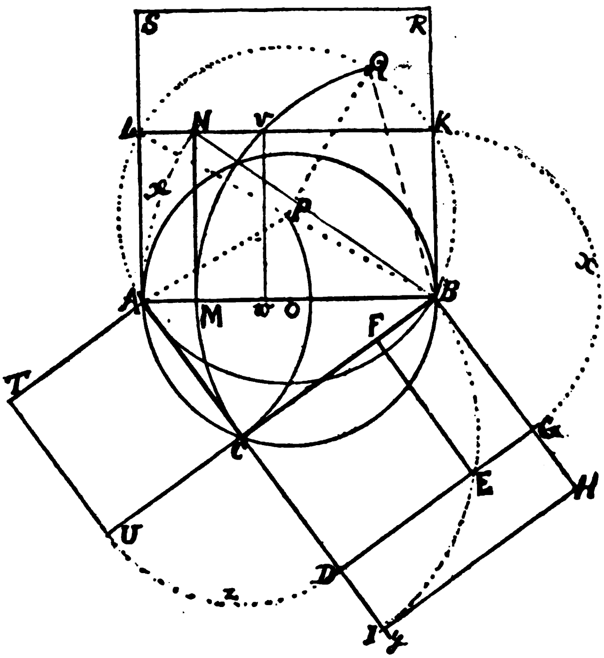


EXHIBIT 13.

24. The triangle $A\ B\ C$ constructed on the diameter $A\ O\ B$ in the circle over $A\ B$ is shown to be a cube triangle; since it is shown, when the hypotenuse $A\ B$ is converted into the tridrant chord $A\ B$ in the greater circle over $B\ P\ L$, then, the major leg $A\ C$ is found to equal the radius $A\ P$. And as it is demonstrated in Section B, Part III, § 6, exhibit 3, that radius, quadrant chord and tridrant chord of any circle are proportioned as sides of the *unit* square, the *dual* square and the *tri* square, and as these three elements compose the triangle's perimeter, it is demonstrated that the squares on the cube triangle's perimeter are proportioned as 1, 2, 3.

The same fact is demonstrable by compasses and ruler; for when the square $A\ U\ C\ T$ is constructed on the minor leg $A\ C$, and when the line $A\ C$ is extended indefinitely to y, and the side $C\ U$, of the unit square of *mg*. 1, is laid off on the line $C\,y$ at D, and when the square $C\ E\ D\ F$ is constructed on $C\ D$, then, the unit square is transposed to a parallel position in the square on the major leg $C\ B$. Now, when the diagonal $C\ E$ is laid off on $C\,y$ at I and a square is constructed on $C\ I$, then it is demonstrated that the square $C\ H\ I\ B$ on the major leg $B\ C$ is the dual square, of *mg*. 2. Finally, when the line $D\ E$ is extended to G and a square is constructed on the hypotenuse $A\ B$, as the square $A\ R\ S\ B$, and when $B\ G$ is laid off on $B\ R$ at K and a line is drawn from K to L, then it can be shown by laying off the major leg $B\ C$ on $B\ A$ at M, that the square $B\ A$ is a trisquare, of the *mg*. 3, since a perpendicular on $A\ B$ at the point M, as the line $M\ N$, marks the prime rectangle $M\ K\ N\ B$ equal to the prime rectangle $C\ G\ B\ D$ or the surd root of a dual square, and by using the *Gf*. $B\ N$, as a radius, from B, it is shown by the curve $N\ x\ A$ that the hypotenuse $A\ B$ is the side of a tri square of *mg*. 3.

PART III.

ANGLES OF THE CUBE TRIANGLE.

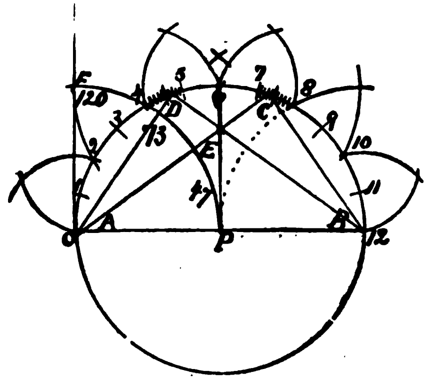


EXHIBIT 14.

25. When a semicircle is divided into 12 equal arcs and each of these arcs are subdivided into 10 minor parts, it is found, that the minor leg of an inscribed cube triangle subtends $4^{7}/_{10}$ of these arcs, while the major leg subtends $7^{3}/_{10}$ of the arcs. Now, when these values are reduced to integral numbers, it may be said, that the arcs subtended by the cube triangles legs are proportioned as 47 : 73; Hence, the semicircumference must be divided into 120 parts, in order to obtain the required angle measure. For it is found, that the acute angles of any right angled triangle inscribed in the semicircle, equal in numeric values, the numeric values of the arcs subtended by the legs. And since the proportioual values 47 and 73 are prime numbers which can not be reduced, it follows, that the lowest numeric value for the right angle contained in a cube triangle, is 120.

Exhibit 14 shows the inscribed triangle *A B C*, the right angle *C*, the greater acute angle *B* and the least acute angle *A*. By transposing the triangle *A B C* to the position *A D B* and by constructing the quadrant *O P E D F O*, we obtain the major acute angle's measure by the arc *P F D* and the minor acute angle's measure by the arc *F E*.

26. Thus it is shown, that commensurational relation naturally exists between segments of a semicircle and angle measure in a quadrant.

THE SQUARING TRIANGLE.

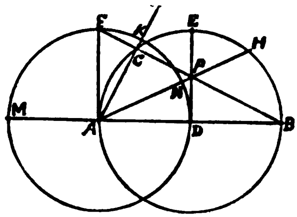


EXHIBIT 15.

27. *B C* is the squaring chord in the circle over the diameter *A B*. *A B C* is the squaring triangle. Now, as the arc *B H E C* is $^6/_{17}$ circumference, the arc *A C* is $^5/_{34}$. These values reduced to a common denominator, become $^5/_{34}$ and $^{12}/_{34}$, the sum of which ($^{17}/_{34}$) gives the semicircumference. Hence, the proportion of the two acute angles are 5 : 12, and the right angle's measure is 17. This is shown by constructing the quadrant *D K E A D* and by drawing another squaring chord as *A P H*. For then, the two acute angles (*A* and *B*) are represented by the angles *D A N* and *D A K*; the former, measures the arc *D N*, the latter, measures the arc *D A K*. Dividing the quadrant arc *D N K E* into 17 equal parts, it is found that *D N* measures 5 and *D N K* measures 12 of these parts. Hence, the angles of the squaring triangle are 5, 12, 17. The squares on the perimeter of this triangle are shown to be of the magnitudes 1, 4, 5; since *A C* : *C B* as 1 : 2, and $A\ C^2 : C B^2 : 1 : 4$; and it follows, that $B A^2 = 5$.

THE RECTIFYING TRIANGLE.

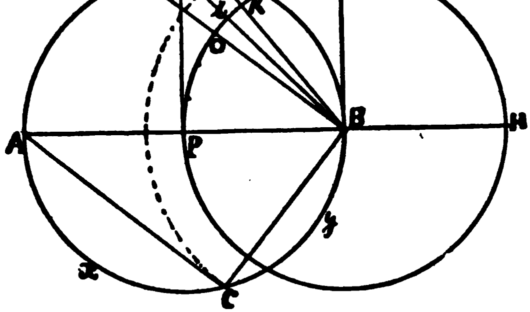


EXHIBIT 16.

28. *B D* is the rectifying chord in the cirle over the diameter *A B*. *A C* is another rectifying chord parallel to *B D*. *A B C A* is the rectifying triangle. The arc *A x C* measures $^2/_7$ circumference, hence the arc *C y B* measures $^3/_{14}$. These values. reduced to a common denominator, become $^3/_{14}$ and $^4/_{14}$, the sum of which($^7/_{14}$) gives the semicircumference. And it follows, that the proportion of the two acute angles are 3:4, and the right angle's measure is 7. This is shown by the quadrant *B P E C' B*, for it is found, when the arc *P O* is trisected and one of these parts is used as a dividing measure for the quadrant arc *P O K M*, seven of these parts are contained in the arc. And when the minor leg *B C* is laid off in the quadrant, from *B C' E*, then the two acute angles are represented by *P B O* and *P B K*. The former measures 3 and the latter, measures 4 of the 7 parts contained in the quadrant arc.

29. The squares on the rectifying triangle are found to be proportioned as 2, 3, 5, which can be demonstrated by compasses and ruler.

30. Summary of the foregoing demonstrations shows:

That squares on perimeter of the *rectifying* triangle are
proportioned as 2, 3, 5;
angles, as 3, 4, 7.

That squares on perimeter of the *squaring* triangle are
proportioned as 1, 4, 5;
angles, as 5, 12, 17.

That squares on perimeter of the *cube* triangle are
proportioned as 1, 2, 3;
angles, as 47, 73, 120.

THE PYTHAGOREAN PROBLEM.

31. This problem is one of the oldest in the history of geometry, aside from the problems: squaring the circle and doubling the cube. The Pythagorean problem involves the solution of triangles, and the first step toward that solution, suggests the doubling of the square which involves a constant ratio of side and diagonal of squares

Constructively, Greek geometers demonstrated that the square on the hypotenuse of a prime triangle equalled, in area, the two squares constructed on the legs of the triangle. They demonstrated with compasses and ruler like this:

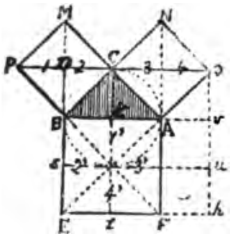


EXHIBIT 17.

32. *A B C* represents the given prime triangle. The squares constructed on the sides of the triangle are: *B M C B* on the leg *B C*, *A N C O* on the leg *C A* and *B A F E* on the hypotenuse *A B*. Now, if the two lines *F A* and *E B*, are extended to *N* and to *M*; and if the line *P O* is drawn through the point *C*, then it is shown, that the two squares on the legs *B C* and *C A* equal the one square on the hypotenuse *A B*, since the lines drawn from *A* to *E* and from *A* to *F* show, that the 8 triangles of the square *A E F B* equal the 8 triangles of the two squares *A N O C* and *A M C B*.

COROLLARY.

33. By constructing the square *D h o E*, it is shown, that in that square is contained the areas of all the squares on the prime triangle's perimeter plus the area of the prime triangle *A B C*.

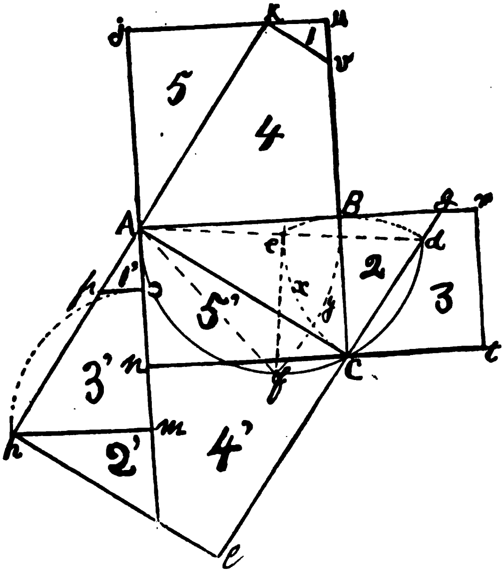


EXHIBIT 18.

34. Other geometers succeeded in various ways to demonstrate the same fact, namely, that the sum of squares on any rightangled triangle equals the square on the triangle's hypotenuse. Exhibit 18 shows the cube triangle *A B C* which is constructed by the tridrant chord *A C*, by the quadrant chord *A f* = *A B*, and by the radius *C e* = *C B*. In the diagram it is shown, that the several parts (2 and 3) composing the square on the leg *B C*, added to the several parts (1, 4, 5) composing the square on the leg *A B*, fit into and are equal to the several parts 1', 2', 3', 4', 5' composing the square on the hypotenuse *A C*.

SCHOLIUM.

35. As it is good disciplinary practice for a pupil to construct thi^ and similar diagrams, direction for the construction thereof may be of some value.

After constructing the squares $B\ C^2$, $B\ A^2$, $A\ C^2$ on the triangle *A B C*, proceed to extend the line *l c* to *g;* extend the line *h A* to *k;* extend the line *j A* to *i;* extend the line *t C* to *n.*

Next, construct a perpendicular line on *i n*, from *h* to *m*. From *m* as a center, with *m h* for radius, describe the curve from *h* to *o*, and construct at the point *o* on *n A* the perpendicular line *o p*. Finally, construct at the point *k* the line *k v* perpendicular to *A k*, then, all the parts are given, which, by transposition demonstrate the fact, that $BC^2 + AB^2 = AC^2$.

THE ARITHMETICAL DEMONSTRATION.

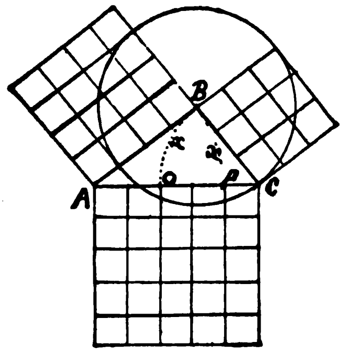


EXHIBIT 19.

36. But mere constructive demonstration did not satisfy science. Mathematicians expected to find numerical figures giving the same result and Pythagores was the first who found the figures 3, 4, 5 which suited the requirements for an arithmetical demonstration. In all probability, he constructed a square and began by dividing the side thereof into a number of equal parts. When he came to the "five" division, he, with compasses, described with the span *A P*, and with the span *C O*, the two curves which intersect each other at *B*. He made *B* the center and described with *B C* the circle which demonstrated to him, that the angle *A B C* was a rightangle, since it divided the circle into four quadrant arcs. Next, he divided the square on the hypotenuse into 25 equal parts, and the squares on the legs into 9 and 16 equal parts, respectively; thus, he had solved the problem arithmetically as well as geometrically, since $9 + 16 = 25$.

A MODERN ASPECT OF THE PYTHAGOREAN PROBLEM.

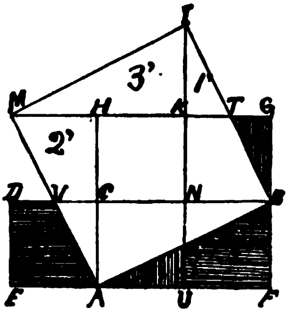


EXHIBIT 20.

37. Analytically examined it is found that the square on th hypotenuse of any rightangled triangle equals four times the given triangle's plane plus one fraction equal in area to a square constructed on the linear difference of the given triangle's legs.

A B C represents the given triangle.

The legs *A C* and *C N B* show, that they are proportioned as 1 : 2.

The squares on these legs are proportioned as 1 : 4, as shown by the squares *C E A D* and *A G H F*. The square on the hypotenuse *A B* is shown to equal the five square units composing the plane of the figure *C D E F G H C*. The diagram further shows, that each of the four triangles *A B C*, *A H M*, *M K I* and *I N B*, contained in the square on *A B*, are equal in area to the center square *C K H N*. Hence, it may be said, that the square on any rightangled triangle's hypotenuse, equals four times the given triangle's plane plus the square on the difference of the triangle's legs, which, in this case, is one unit.

38. The constant ratio of side and diagonal of squares, directly associated with the doubling of a square, can only be demonstrated satisfactorily by commensurational arithmetic which is explained and applied in Section C.

THE ARCHIMEDIAN PROBLEM.

39. The Archimedian Problem is another important discovery dating back 2500 years. It involves the relation of cones, spheres and cylinders of equal altitude, and demands a process which will demonstrate the fact, that a cone, a sphere and a cylinder of equal altitude are proportioned in volumes as 1, 2, 3. The geometric demonstration of this problem which constitutes the basis for the arithmetical computation, is shown by the Exhibit 21.

The arithmetical computation will be given in Section C.

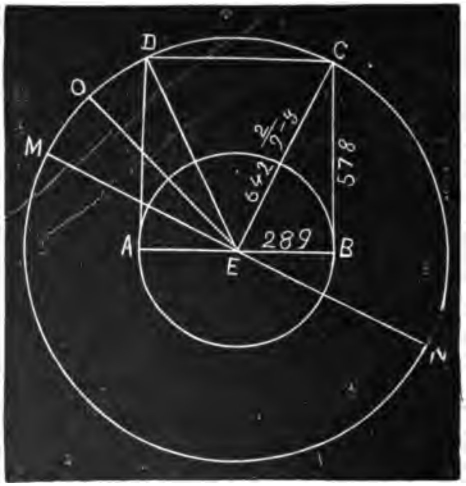


EXHIBIT 21.

40. The diameter *A E B* represents the given altitude of a sphere the great circle of which is represented by the circle over the diameter *A E B*. The square *B D C A* represents the vertical section of a cylinder of equal altitude of the given sphere's altitude *B E A*. The triangle *E D C E* represents the vertical section of a cone, the altitude of which equals the altitude of the giving sphere and cylinder. The circle over *B E A* represents the common base for all three volumes. Now, it is found, when the slant of the cone (*E C* or *E D*) is made a radius and a circle is described from *E*, as the circle over the diameter *N E M*, that this greater circle is proportioned to the first given circle over the diameter *B E A* as 5 : 1.

For it is shown that $B\,C = 2\,(B\,E)$, hence, the square on $B\,E$ is proportioned to the square on $B\,C$ as 1 : 4; and it follows, that the square on $E\,C$ (which is the hypotenuse of the rightangled triangle $E\,B\,C\,E$) is of the magnitude 5. And because, the radius $E\,N$ of the greater circle equals the slant $E\,C$, it follows: that radius $B\,E$ is to radius $E\,N$ as $.\sqrt{1} : .\sqrt{5}$; and it further follows: the minor and the major circles, given in the diagram, are proportioned in magnitudes as 1 : 5, since circles are to each other as the squares on their diameters.

To obtain the curved superfacies of the given cone, bisect the arc $D\,O\,M$ at O, then the arc $M\,O$ is found to be one twentieth of the major circle's circumference, and consequently the sector $O\,E\,N\,C\,D\,O$ measures an arc equal to $^9/_{20}$ of the major circle's circumference, which arc is found when the sector is cut out and formed into a cone, fits the circle over the diameter $B\,E\,A$.

SUMMARY OF PART III, SECTION B.

MATHEMATICAL EVOLUTION OF MAGNITUDES.

Basis of Trigonometry.—The Geometric Canon of Proportion.

THE SOLVING TRIANGLES.

Their Properties and Use.

THE PYTHAGOREAN PROBLEM.

THE ARCHIMEDIAN PROBLEM.

SUPPLEMENT.

SUGGESTION TO THE TUTOR.

Part III presents a large field for practice with compasses and ruler, and especial attention should be given to the use of the *Gf.* in demonstrations where planes of given figures can be proven commensurately related, by compasses and ruler. But, in no case, should these demonstrative exercises be performed outside the field of a circle; so as to keep in mind, that every geometric quantity is contained in and is a part of the circle. In connection with these exercises it is well by way of review and for freshening the memory to have the pupil name the several parts and give reasons why this or that part of a given figure is so and so related to the circle.

Since the main object aimed at in a course of geometric study is the cultivation of the reflective and reasoning faculties, pupils should be given opportunities to demonstrate by new methods of their own that they fully understand what has been taught them.

SECTION B

PART IV

SECTION B.

Part IV.

FUNDAMENTAL PROPOSITIONS, THEOREMS AND FORMULAS.

LEMMA 1.

1. Every triangle's perimeter is composed of three lines: The longest, the shortest and the connecting line, except the trihedron, the sides of which are equal.

PROPOSITION 1.*

2. The longest line in the perimeter of any triangle is shorter than the other two.

THEOREM 1.

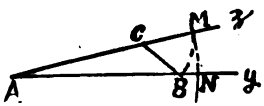


EXHIBIT I.

3. *A B C A* is a given triangle's perimeter. The shortest line in that perimeter is *B C*. The longest line in that perimeter is *A B*. The connecting line is *A C*. It is evident to the eye that the longest line *A B* is shorter than the other two, *B C* and *C A*.

* The first proposition is called the "pons asinorum," the Greek term for "asses' bridge." It is said to have originated with a playwright named Aristophanus, living in Athens. Greece, about 420 B. C. He ridiculed the geometers of the period by declaring he know a donkey possessed of extraordinary mathematical knowledge and invited the learned people to come and see. A triangle was marked out on the stage and a starved donkey was tied to a stake at one extremity of the triangle's longest line, while at the other end of the line a whisk of hay was fastened. On releasing the hungry animal the donkey went directly along the line leading to the food, which action on the part of the donkey the actor claimed, much to the amusement of the audience, was evidence that his donkey knew the longest line in the figure was shorter than the other two.

DEMONSTRATION.

4. Extend $A\ C$ to z. Lay off $C\ B$ on $C\ z$ from C to M. Then it is shown, that $A\ M = A\ C + C\ B$. Extend $A\ B$ to y. Lay off $A\ M$ on $A\ y$ from A to N. Then it is shown, that the longest line ($A\ B$) is shorter than the other two ($A\ C$ and $C\ B$); since $A\ N = A\ C + C\ B$ and $A\ N$ is longer than $A\ B$ as shown by $B\ N$, (the difference.)

ALGEBRAICALLY EXPRESSED.

By substitution: $l = A\ B.$ $s = B\ C.$ $c = A\ C.$ $d = B\ N.$

1st Formula: $l < s + c.$

2nd Formula: $s + c = l + x.$

3rd Formula: $x = d.$

$$\overline{c + s - d} = l.$$

$$l + d = c + s.$$

LEMMA 2.

5. Every prime segment is bounded by one uniform curve and one straight line. (See § 15, Section A, Part I, page 4.)

PROPOSITION 2.

6. Every arc in any circle is longer than the arc's subtending chord.

THEOREM 2.

EXHIBIT 2.

7. $a\ d\ c\ e\ b\ a$ is a given segment. $a\ d\ c\ e\ b$ is the arc; $a\ b$ is the chord subtending that arc. It is evident to the eye, that the distance $a\ d\ c\ e\ b$ is longer than the distance $a\ b$. The difference in length is represented by $b\ n$.

COROLLARY.

8. As it is shown by theorem 1, that two sides of any triangle is greater in extent than the third, it follows, that a triangle constructed in any segment, as shown by the lines *a c* and *a b*, demonstrates that the arc is greater in extent than its subtending chord, since the distance *a c* and *c b* are greater than the distance *a b*, and since triangles inscribed in the lesser segments *a d c a* and *c e b c*, show, that the curves *a d c* and *c e b* are longer in extent than the distances *a c* and *c b* which already are proven to be longer than the line *a b*, the subtending chord.

ALGEBRAICALLY EXPRESSED.

By substitution: $a\,d\,c\,e\,b = a.$

$$a\,b = c.$$

$b\,n = x$ (the difference of c and a).

4th Formula: $a > c$

$$\underline{\quad c < a.}$$

5th Formula: $\overline{a - x} = c.$

$$\overline{c + x} = a.$$

6th Formula: $\dfrac{a}{c} = 1 + x.$

$$a - c = x.$$

PROPOSITION 3.

9. In every circle there is a segment so proportioned in chord and arc that the difference is 1. That segment is technically called the *pi* segment.

THEOREM 3.

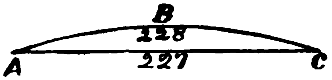


EXHIBIT 3.

10. Let two lines be so divided into equal parts by a common measure that the lesser of the two lines measures 227 of these parts and the greater measures 228 of the given parts, then it is

found, when the longer line is uniformly curved into an arc sub-tended by the shorter line, as a chord, that 908 of these segments formed into an endless uniform circuit exhibit the circumference of a circle and the perimeter of an inscribed regular polygon of 908 sides, each of which sides measures $^1/_{189}$th part of the circle's diameter which also is the diameter of the inscribed polygon.

ALGEBRAICALLY EXPRESSED.

11. By substitution:

a = arc $A\ B\ C$. c = chord $A\ C$. 1 = difference of mg.

7th Formula:

$$\frac{a}{c} = 1 + x. \quad x = 1.$$

$$a - c = 1.$$

$$\frac{c + 1}{a} = 1.$$

$$a - 1 = c$$

$$x + c = a. \quad a = c + 1.$$

PROPOSITION 4.

12. In any given sector the segment's altitude is the versed sine of one-half the segment's arc, and one-half the segment's chord is the right sine of the same arc, hence it follows, that the altitude of any given sector's angle plane is the co-sine of one-half the sector's arc.

THEOREM 4.

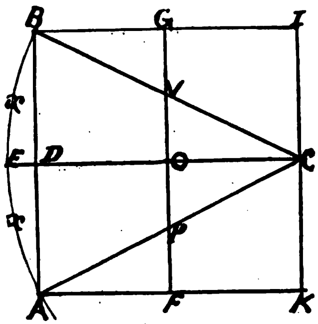


EXHIBIT 4.

13. $C\ A\ E\ B\ C$ is the given sector, $A\ E\ B$ the arc, $A\ D\ B$ the chord. $D\ E$ is the altitude of the given segment and represents the vs. of the arcs $E\ x\ B$ and $E\ x\ A$, while $D\ A$ and $D\ B$ represent, each, the RS. of the same arcs. $D\ C$ is the altitude of the sector's angle-plane and represents the cs. of either arc, $E\ x\ A$ or $E\ x\ B$. $C\ D + D\ E$ is radius of the circle of which the given sector $C\ A\ E\ B\ C$ is a part.

8th Formula:

14. By substitution: let $a = A\ x\ E\ x\ B$ the arc;
let $c = A\ D\ B$ the chord;
let $l = O\ C\ P$ the angle.

Then: $\overline{\text{R}-\text{vs.}} = \text{cs.}\ (D\ C.)$

$$\frac{c}{2} : \frac{a}{2} :: \text{RS.} : \frac{a}{2}$$

$$a : c :: \text{RS.} : l.$$

PROPOSITION 5.

15. The angle plane of any sector equals a rectangle plane composed in perimeter of the sector's chord and twice the altitude of the angle plane; or, a rectangle plane composed in perimeter of the altitude of the angle plane and twice the sector's chord.

THEOREM 5.

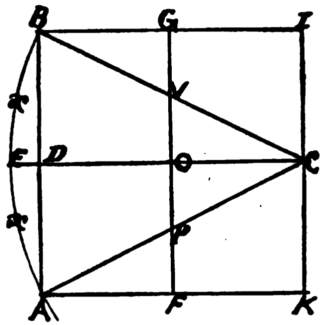


EXHIBIT 5.

16. $C\ B\ D\ A\ C$ is the given angle plane. $B\ C\ I\ D$ represents the rectangle of equal area to the given sector, since $B\ I\ C\ B = A\ D\ C\ A$, and the rectangle's perimeter is composed of $\overline{B\ D + I\ C = B\ D\ A}$ the chord, while, $\overline{D\ C + B\ I = 2\ C\ D}$ or twice the altitude $C\ D$.

17. Likewise, the rectangle $B\ F\ G\ A$ represents the rectangle of equal area to $C\ B\ A\ C$, since $O\ C\ V = G\ B\ V$ and $C\ O\ P = P\ A\ F$. The perimeter is shown to be composed of $\overline{B\ G + A\ F} = D\ C$ the altitude, while $\overline{G\ F + B\ A} = 2\ A\ B$ or twice the chord $A\ B$.

9th Formula:

18. Let a = altitude, let b = chord, let c = angle plane.

Then: $\frac{a}{2} \times b = c$.

$$c = a \times \frac{b}{2}$$

LEMMA 3.

THE IDIOSYNCRASY OF MATHEMATICS.

19. It is stated in Section A, Part III, page 3, that a globe contains relatively to its superfacies more volume than any other form. Likewise, the circle contains more area under its given boundary than the same boundary cast into any other form, can contain. If therefore, the circumference of a circle or any part thereof is rectified into a line and its numeric magnitude is multiplied by the numeric magnitude of the circle's radius or any part thereof, the product obtained, though alike in numeric value, does not represent an area equal to the area contained under the same boundary in the curved form. Hence, when a curved boundary is rectified, *decrement* of plane follows and a corresponding linear *increment* is necessary to balance the difference occasioned by the change of form. This has been called the idiosyncrasy of mathematics and to the unexplained solution of this peculiar problem is due the apparent conflict of geometry and arithmetic, which, however, readily yields to adjustment as will be shown by commensurational arithmetic in Section C.

PROPOSITION 6.

20. The plane of any sector is contained in a rectangle, composed in perimeter of the sector's rectified arc and two radii plus commensurate linear *increment* required to balance the areal *decrement* of plane resulting from rectification of the curve.

THEOREM 6.

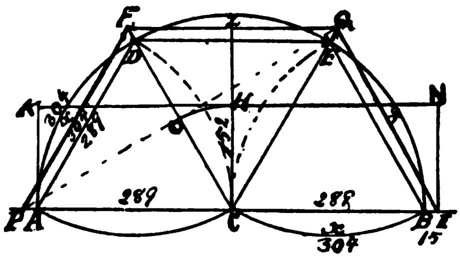


EXHIBIT 6.

21. *A B* is a given circle's diameter, *C* is the circle's center. *A D E B* represents one-half perimeter of an inscribed hexagon of which three sextants are given, as *A C D x A*, *D C E z D*. *E C B y E*. *P F G I* represents the given semicircumference cast into a polygon form parallel to the given inscribed hexagon. Hence, *P F* and *I G* represent two rectified arcs of the three given sextants. The extensions *P A* and *B I* added to the radii *C A* and *C B* show the diametric increment required to balance the areal decrement resulting from conversion of the circle curve to the angular perimeter. Now, when the line *C F*, which equals the rectified arc *P F*, is bisected at *O* by drawing a line from *G* to *P*, then, the distance *C O* represents one-half the rectified arc of a sextant. Finally, lay off on the radius *C z*, the distance *C O* from *C* to *H*, and construct the rectangle *H A K C*, which in numeric value equals the numeric value of the given sextant.

22. But, it will be shown, hereafter, that six of these rectangles do not contain as much area as the equal square to circle obtained by the squaring process described in Section B, Part II. exhibit 13. § 34. If, on the contrary, the diametric increment

($B\ I$) is added to the radius $C\ B$ and a rectangle is constructed as $C\ N\ H\ I$, that greater rectangle equals in area the plane of the given sector $C\ E\ y\ B\ C$, and it is found that six such rectangles contain an area equal to the circle plane converted into a square. Thus, the proposition is exemplified geometrically, since the greater rectangle $C\ H\ N\ I$ is composed in its perimeter of 2 (R + $B\ I$) and the sector's rectified arc.

PROPOSITION 7.

23. The difference in the plane of a rectangle formed of radius and rectified circumference and the plane of a square equal to circle, amounts to a fraction of space proportioned to the circle's whole plane as the *pi* fraction is proportioned to the whole *pi* value

THEOREM 7.

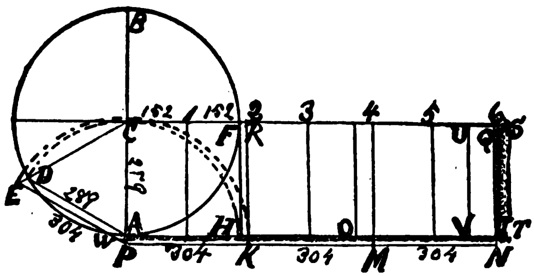


EXHIBIT 7.

24. A quartered circle is given over the diameter $A\ C\ B$. Six rectangles, as $A\ H\ K\ C$ (of exhibit 6) are constructed into the rectangle $A\ Q\ I\ C$ (exhibit 7). On the radius $C\ A$ is constructed the square $A\ F\ C\ H$, and it is found, that three R^2 + the fraction $U\ I\ V\ Q$ equal the six rectangles composing the plane $A\ Q\ I\ C$. $P\ E^2$ represents the rectified sextant arc^2, or twice the rectangle $C\ N\ H\ B$ (of exhibit 6). Now, if $P\ E$ is laid off on a line from P to N parallel to the line $A\ I$, and the square $P\ K\ R\ C\ A$ is formed then, the relative proportion of R^2 to the square on the

rectified sextant's arc and three such squares compose the rectangle $P\,Q\,N\,C$ which in area equals the area of the given circle. The decrement of area resulting from the rectification of the semicircle's curve is represented by the space $A\ N\,P\,I$, or by the space $I\,S\,Q\ T$, either of which represent portions of space proportioned to the whole area $C\,P\,N\,Q$, as the *pi*-fraction is proportioned to the whole *pi*-value—which will be demonstrated later on in Section C.

25. But in as much, as both the given rectangles $C\ I\ Q\ A$ and $C\ N\ Q\ P$ represent areas under *rectified* arcs, and while the common increment aud decrement adjusting these rectangles' perimetric relations are pointed out, still another adjustment is necessary to show the mutual relations of these perimetric boundaries to the curved boundary of the circle's circumference.

This adjustment is found in a mean-proportional square between the square on $E\ P$ and $A\ D$; that is, a *mp.* ² of arc ² and chord ² of any sextant.

The increment and decrement, thus given in elbow-form, is shown by the shaded space from W to I and from I to 6, in the exhibit 7.

10th Formula:

26.
$$\frac{\odot}{3} = \overline{mp.^2 \text{ of } R^2 \,\&\, \left(\frac{\bigcirc}{6}\right)^2}$$

$$\overline{R \times \frac{\bigcirc}{2}} = \odot$$

By substitution: *dec.* for decrement, *inc.* for increment, *rct.* for rectified, *fct.* for fraction.

Then: $\overline{R \times rct. \frac{\bigcirc}{2}} = \odot - dec.$

$$\odot = \overline{R \times rct. \frac{\bigcirc}{2} + inc.}$$

$$dec. \ . \odot : : \pi\, fct\ : \pi.$$

$$\pi\, fct. : \pi : : inc. : \overline{R \times rct. \frac{\bigcirc}{2}}$$

LEMMA.

27. A rectangle is converted into a square by rectifying the minor and the major sides of the rectangle into one line, by describing a semi-circle on that line as over a diameter, and by constructing a right sine in that semicircle, so that, the minor side of the given rectangle equals the versed sine to the arc which is subtended by the constructed right sine. The rightsine thus obtained is found to be the side of a square equal in area to the given rectangle. (See the canonic axiom, § 15, Section B, Part II.)

PROPOSITION 8.

28. Equal square to circle equals three tridrants, of the circle, converted into a trisquare.

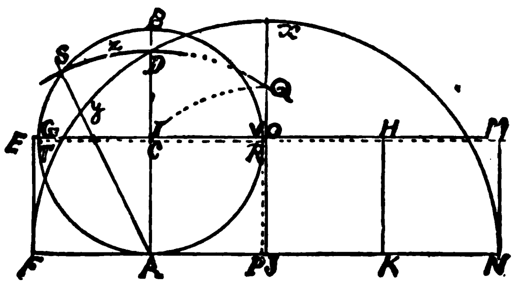


EXHIBIT 8.

THEOREM 8.

29. A quartered circle is given, as the circle *A T B R A*. The squaring chord *A S* represents the side of equal square to circle R^2 is represented *C P R A*, and the greater square *I J O A*, represents the $\overline{mp.^2 \text{ of } R^2 \,\&\, \left(\frac{\bigcirc}{6}\right)^2} = \frac{\odot}{3}$. The rectangle *I N M A* is composed of $3\left(\frac{\odot}{3}\right)$ and represents the area of the given circle in the form of a rectangle, while three times the elbow-form *C I O R P J* represents the adjusting increment to the rectangle composed in perimeter of rectified semicircumference and radius. The proposition calls for a demonstration which converts the given

rectangle $I\ N\ M\ A$ into a square the side of which shall equal the squaring chord $A\ S$. The line $N\ F$ represents the minor and the major side of a given rectangle rectified into a line, the center of which is represented at the point J. From J as a center, describe over the line $F\ N$ the semicircumference $N\ x\ F$, and extend the line $A\ I$ to the point D on the circumference, then, the right sine $A\ D$ is given and is shown to subtend the arc $D\ y\ F$ to which the distance $A\ F$, of the radius $F\ J$, is the versed sine. Now, by using the sine $A\ D$ for a radius, from A as a center, it is shown by the curve $D\ zS$, that the sine $A\ D$ equals the squaring chord $A\ S$. And it is further shown by application of the Gf, that $A\ D^2$ is a tri-square, since the diagonal $J\ I$ laid off on the extension $O\ x$ at Q, marks the side of a dual square from J to Q, and a diagonal from A *to* Q laid off on the diameter $A\ B$, marks, at D, the tri-square. Hence the proposition is verified by construction.

30. 11th Formula:

$$R^2 : \frac{\odot}{3} :: 3\,(R^2) : \odot$$

$$\overline{R^2 \times \pi} = 3\left(\frac{\odot}{3}\right)$$

$$3\,(R^2 + \textit{inc.}) = \odot$$

$$3\left(\frac{\odot}{3}\right)^2 = \overline{\blacksquare = \odot}$$

$$\overline{R \times \textit{rct.}\ \frac{\bigcirc}{2}} + \textit{inc.} = (\blacksquare = \odot)$$

$$\overline{R \times \frac{\bigcirc}{2}} = \odot$$

$$\odot = \frac{\bigcirc}{4} \times \textit{Dmt.}$$

$$R \times \textit{rct.}\ \frac{\bigcirc}{2} : \odot :: R : \frac{\bigcirc}{6}$$

$$\overline{R \times \textit{rct.}\ \frac{\bigcirc}{2}} : \overline{R \times \frac{\bigcirc}{2}} :: R : \frac{\bigcirc}{6}$$

$$\odot - \textit{dec.} : \odot :: R : \frac{\bigcirc}{6}$$

PROPOSITION 9.

31. The square on radius of any given circle is proportioned to equal square to circle as diameter is proportioned to circle's circumference.

THEOREM 9.

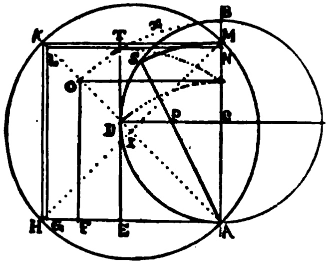


EXHIBIT 9.

32. Given: The circle over diameter *A B.*

Bisect radius *C D* at *P* and draw the squaring chord *A S*, and with that chord, construct on the diameter *A B*, the square *A K M H*, which square represents the equal square to circle over *A B.* On radius *A C* construct the square *A D C E* and form the surd root *A M T E.* Evolve from the R^2, by the *Gf.*, the tri-square *A N L G* which represents 3 (R^2). Thus, the elbow-form *G H K L M N* is given, the plane of which represents the x value $= \pi$ faction; while the corresponding y value is represented by the linear fraction *G H* or *M N.* Now, in as much as the square on radius is proportioned to the equal square as 1 : 3 + π fraction, it follows, that the proposition is verified by construction, since the diameter of any circle is proportioned to circumference of circle as 1 : 3 + π fraction.

12th Formula:

33. Hence the equations:

$$R^2 : \overline{\blacksquare = \odot} : : Dmt. : \bigcirc$$

and

$$\overline{R^2 \times \pi} = \odot$$

COROLLARY.

34. By drawing the curve $O\,x\,B$ it is shown, that the diameter of the given circle is proportioned to the diameter of the circle circumscribing the equal square to circle as $2 : \pi$; and it is shown that the inscribed square of the given circle is proportioned to square on given circle's diameter as $2 : 4$. Hence it follows, that inscribed square in circle is proportioned to equal square to circle as $2 : \pi$, and circle is proportioned to square on diameter as $\pi : 4$. Hence it may be said by

13th Formula:

35.
$$R^2 : \frac{\odot}{4} :: 4 : \pi$$

$$\frac{\odot}{4} : \frac{R^2}{2} :: \pi : 2.$$

FINAL GEOMETRIC DEMONSTRATION.

36. The final geometric demonstration shows by construction, the common increment and decrement equal to the π fraction in a square form.

The demonstration shows, that the equal square to circle contains more area under its square perimeter than the rectangle formed by the rectified semicircumference and the circle's radius, or, the rectangle which is formed of the rectified quadrant arc and the circle's diameter, while, at the same time it is shown, that the square perimeter measures less than either of the given rectangle's perimeters. These facts and the foregoing illustrations relative to natural increment and decrement accounts for *apparent* incommensurable relationship of geometric elements.

DEMONSTRATION.

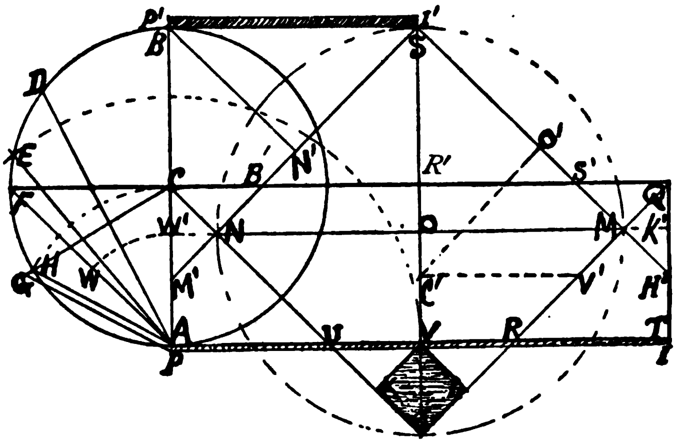


EXHIBIT 9.

37. *A C B* is the diameter of a given circle. *A F* represents the quadrant chord. *A E* represents the rectifying chord. *A H* represents the sextant chord and *P G* represents the rectified sextant arc. *A D* represents the squaring chord.

On a line extended indefinitely from the point *A*, perpendicular to *A B*, lay off two rectifying chords, as *A V* and *V T*, equal to the given circle's rectified semicircumference. From the circle's center *C*, construct another line, as *C Q*, equal in extent to *A T*, perpendicular to *A B* and parallel to *T A*. Then join the points *Q* and *T* by the line *Q T*, which gives the rectangle *A Q C T*. Now, because, in the circle form, the linear magnitudes of radius and semicircumference multiplied together, give a product which numerically expresses the areal magnitude of the circle's plane, it would follow, if change of form did not affect the relation of boundaries and planes, that the rectangle *C T Q A* of equal numeric linear values, should contain as much area as the circle, since the numerical product obtained by the process of multiplica-

tion is the same. This, however, is not the case, as already has been shown in foregoing theorems and will be shown in this in a different and conclusive manner.

38. From the point *V*, construct a line of indefinite extent, parallel to *A B* and perpendicular to *A T*, and beyond the points *J* and *I'*. Bisect the quadrant chord *A F* at *W* and lay off on the radius *C A* the distance *A W* equal to *A W'*. Next proceed to draw the line *W K'* parallel to *A T* and perpendicular to *A C*. Then the central point *O* is given. From *B* construct the perpendicular *B S* parallel to *A V* and equal in extent to *A V*. From *O*, as a center, with *O S* as radius, describe a circle which marks the intersecting points *M*, *N*, *J*, *S*. which points also quarter the circumference of the circle over the diameters *S J* and *N M*. Now inscribe the square *N M S J*, then it is found that the side of this inscribed square equals the squaring chord *A D*. Hence, the square *N S M J N*, is shown to be the equal square to the first given circle over the diameter *A B*.

39. The diagram shows the rectangle *B V S A* besides the rectangle *C T Q A*, both of equal area, and both showing the common increments (*A I T P* and *B I' S P'*). If a line is constructed from *B* to *N'* perpendicular to *N S*; and if the lines *J N* and *S N* are extended to *C* and *M'*; and if the lines *S M* and *J M* are extended to *H'* and to *Q*, then, it can be shown by transposition of several parts, that each of the given rectangles, equal the square *S J N M* minus the square fraction *V J K L*, which square equals either of the two given increments *P' S I' B* and *A I P T*.

40. For it is found; that *N' B S* = *S C' O' S*. That *B N' N C B* = *O' M V' C' O'*. That *U A M' N U* = *C' V' R V C'*.

Hence, there is nothing left of the rectangle $B\ V\ S\ A$ excepting the two triangular fractions $C\ N\ B'$ and $C\ N\ M'$ which fit the spaces $U\ V\ K$ and $V\ L\ R$. And it follows that the square $K\ L\ V\ J$ represents the required increment or the decrement resulting from conversion of the circle curve.

41. Likewise, when the triangle $Q\ R\ T\ Q$ and the triangle $C\ U\ A\ C$ are transposed to the spaces $S\ R'\ B'\ S$ and $S\ R'\ S'\ S$, then, there are only left in the rectangle $C\ T\ Q\ A$, the two triangular fractions $C\ N\ B'$ and $Q\ M\ S'$, which two fractions fit the spaces $V\ L\ R$ and $U\ K\ V$. Thus, in either case, the discrepancy between arithmetic and geometry is pointed out by the π fraction in square form.

SUMMARY OF PART IV, SECTION B.

NINE FUNDAMENTAL PROPOSITIONS.

10 THEOREMS—13 FORMULAS.

IDIOSYNCRASY OF MATHEMATICS.

Natural Increment and Decrement.

HARMONIZED RELATION OF GEOMETRY AND ARITHMETIC.

SUPPLEMENT.

SPECIAL SUGGESTIONS TO THE TUTOR.

The author's experience among tutors while compiling the "new geometry" disclosed the fact that not many teachers of mathematics understand the elementary principles of geometry pure and simple. He found many who mistake the *application of geometry*, to mensuration and to physics, for the science itself. It is therefore proper, here, at the end of Section B, to draw attention to certain pertinent facts before entering upon the study of the principles' application to computation generally the study of which begins in Section C.

Some teachers doubt their own ability to draw a circle and locate a center, or to draw a line and divide a circumference into equal parts; for the author has on many occasions been asked to "demonstrate and prove that this or that given chord subtends this or that given arc" after *rules have been given for the construction* of the chord. The cause of questions like this, coming from intelligent teachers, is perhaps traceable to the confused arrangement of geometric elements and fundamental principles in old-fashioned text-books on geometry. Since Descarte's analytics and Legendre's sophisticated reasonings about incommensurable quantities have become part of the so-called "higher" mathematics, many tutors believe that pure geometry is a difficult and unprofitable study which can not be taught or acquired without tautalogical syllogisms. Yet, no study is more simple, exact and profitable than elementary geometry, the demonstrations of which require no other proof than that furnished by compasses and ruler dexterously handled with reasonable care acquired by practice. But,

fixed rules must be complied with in construction, and the same result must be obtained by similar operations in a number of circles, each greater than the preceding one. (See 37–38, page 10, Section A, Part I.)

Before passing from the study of pure geometry based on demonstrations with compasses and ruler and on object lessons, it is well for the tutor to explain to his class the relation of pure geometry to commensurational arithmetic. Point out that all the geometric elements and forms which the pupils have acquired knowledge of as so many geometric quantities expressed in dots, lines and curves are now to be expressed in fitting numeric figures obtained by commensurational arithmetic. Hence, it may be said that commensurational arithmetic is, *geometry translated and expressed in numbers*. It should be understood that geometry thus translated into numbers becomes the *science of ratios;* and when a sufficient number of exact and finite ratios are obtained and tabulated, these tables and their measures of proportion, become economic instruments in computation which can be applied to conventional mensuration and to all branches of physics.

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